Formal Theories of Predication,

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1. Introduction

A formal ontology, we have said, is based on a theory of predication and not on set theory as a theory of membership. Set theory, of course, might be used as a model-theoretic guide in the construction of a theory of predication; but such a guide can be misleading if we confuse membership with predication.

A theory of predication depends on what theory of universals it is designed to represent, where, by a universal we mean that which can be predicated of things. A universal is not just an abstract entity, in other words, but something that has a predicative nature, which sets do not have.

Our methodology in studying a theory of predication is to reconstruct it as a second-order predicate logic that represents the salient features of that theory.

By a second-order predicate logic we mean an extension of first-order predicate logic in which quantifiers can reach into the positions that predicates occupy in formulas, as well as into the argument positions of those predicates.

This means that just as the quantifiers of first-order logic are indexed by object variables, which are said to be bound by those quantifiers, so too the quantifiers of second-order logic are indexed by predicate variables, which are said to be bound by those quantifiers.

We need only add n-place predicate variables, for each natural number n, to the syntax for first-order logic. We will use for this purpose the capital letters F^n, G^n, H^n, with or without numerical subscripts, as n-place predicate variables; but we will generally drop the superscript when the context makes clear the degree of the predicate variable. Sometimes, for relational predicates, i.e., where $n > 1$, we will also use R and S as relational predicate variables as well.

2. Logical Realism

The axioms described so far apply to nominalism and conceptualism, as well as to logical realism. What distinguishes logical realism is an axiom schema that we call a *comprehension principle*, (CP).

$$\text{(CP)} \qquad (\exists F^n)(\forall x_1)...(\forall x_n)[F(x_1,...,x_n) \leftrightarrow \varphi],$$

where φ is a (second-order) formula in which F^n does not occur free, and $x_1, ..., x_n$ are pairwise distinct object variables occurring free in φ.

What the comprehension principle does in logical realism is posit the "existence" of a universal corresponding to every second-order formula φ.

This is so even when φ is a contradictory formula, e.g., $\neg[G(x) \rightarrow G(x)]$. In other words, contrary to what has sometimes been held in the history of philosophy, there are properties and relations, that on logical grounds alone cannot have any instances. Such properties and relations cannot be excluded without seriously affecting the logic of logical realism. In particular, because we cannot effectively decide even when a first-order formula is contradictory, the logic would then not be recursively axiomatizable.

The universal instantiation law for properties and relations is stated as the following theorem schema:

$$(\text{UI}_2) \qquad (\forall F)\psi \rightarrow \psi[\varphi/F(x_1,...,x_n)],$$

where φ can be properly substituted for F in ψ. This law is equivalent to the comprehension principle.

The contrapositive of (UI_2) is the second-order law for existential generalization,

$$(\text{EG}_2) \qquad \psi[\varphi/F(x_1,...,x_n)] \rightarrow (\exists F)\psi.$$

Second-order predicate logic is essentially incomplete with respect to the so-called "standard set-theoretic semantics" for predicate quantifiers.

But, as we have already noted (in our 2nd lecture), whether or not that incompleteness applies to the theory of universals that is the basis of a second-order predicate logic is another matter altogether.

Note: Just as the first-order axiom

$$(\exists x)(a = x),$$

makes our existential presuppositions explicit for singular terms, so too (CP) makes explicit our ontological commitment to the universals in question.

Note: In the ontology of natural realism, the assumption that a natural property or relation corresponds to any given predicate or formula is at best an empirical hypothesis. Whether or not a given predicate or open formula stands for a natural property or relation, in other words, cannot be settled by logic alone. The comprehension principle (CP), in other words, is not a valid thesis in **natural realism**. We will forego giving a fuller analysis of natural realism at this point, however, until a later lecture when we consider conceptual natural realism.

The status of the comprehension principle, (CP), as these remarks indicate, is an important part of the question of what metaphysical theory of universals is being assumed as the basis of our logic as a formal theory of predication.

3. Nominalism

The basic thesis of nominalism is that there are no universals, and that there is only predication in language. This suggests that the comprehension principle (CP) must be false, which is why the formal theory of predication that is commonly associated with nominalism is standard first order predicate logic with identity. In fact, however, the situation is a bit more complicated than that.

It is true that according to nominalism first-order predicate logic gives a logically perspicuous representation of the predicative nature of the predicate expressions of language.

> Nominalism: The logico-grammatical roles that predicate expressions have in the logical forms of first-order predicate logic explains their predicative nature.

Predicate constants are assigned the paradigmatic roles in this explanation.

But that does not mean that predicate constants are the only predicative expressions that must be accounted for in nominalism. In particular, any open first-order formula of a formal language L, relative to the free object variables occurring in that formula, can be used to define a new predicate constant of L. Such an open formula would constitute **the definiens of a possible definition** for a predicate constant not already in that language.

An open formula, accordingly, must be understood as implicitly representing a predicate expression of that formal language.

Potentially, of course there are infinitely many such predicate constants that might be introduced into a formal language in this way, and some account must be given in nominalism of their predicative role.

Question: How can nominalism represent the predicative role of open first-order formulas?

Now an account is forthcoming by extending standard first order predicate logic to a second order predicate logic in which predicate quantifiers are interpreted **substitutionally**.

That is, we can account for all of the nominalistically acceptable predicative expressions of an applied first-order language without actually introducing new predicate constants by turning to a second order predicate logic in which predicate quantifiers are interpreted substitutionally and predicate variables have only first-order formulas as their substituents.

Now there are constraints that such an interpretation imposes, and as we have shown elsewhere those constraints are precisely those imposed on the comprehension principle in standard "predicative" second-order logic.

The use of the word 'predicative' here is based on Bertrand Russell's terminology in *Principia Mathematica,* where the restriction in question was a part of his theory of ramified types.

Apparently, because of the liar and other semantical paradoxes, Russell, despite his being a logical realist at the time, thought that only the so-called "predicative" formulas should be taken as representing a property or relation.

The restriction, simply put, is that no formula containing bound predicate variables is to be allowed in the comprehension principle for nominalism, which can be formulated as follows:

$$\text{(CP!)} \qquad (\exists F^n)(\forall x_1)...(\forall x_n)[F(x_1, ..., x_n) \leftrightarrow \varphi],$$

where φ is a formula in which (1) no predicate variable has a bound occurrence, (2) F^n does not occur free in φ, and (3) $x_1, ..., x_n$ are pairwise distinct object variables occurring free in φ.

Under a substitutional interpretational the appearance of an existential posit regarding the "existence" of a universal in the quantifier prefix $(\exists F^n)$ is just that, an appearance and nothing more.

In an applied formal language, the principle (CP!) involves no ontological commitments under such an interpretation beyond those one is already committed to in the use of the first-order formulas of that language.

By interpreting predicate quantifiers substitutionally, in other words, (CP!) will not commit us ontologically to anything we are not already committed to in our use of first-order formulas, and, as we have said, it is the logico-grammatical role of predicate expressions in first-order logic that is the basis of nominalism's theory of predication.

Note: Corresponding to the restricted comprehension principle (CP!), only a restricted version of the universal instantiation law is provable in nominalism.

That is, if no predicate variable has a bound occurrence in φ, then, the following is provable in nominalism,

$$\text{(UI!}_2\text{)} \qquad (\forall F)\psi \rightarrow \psi[\varphi/F],$$

where φ can be properly substituted for F in ψ. From (UI!$_2$), we can then derive the restricted version of existential generalization for predicates.

> What these observations indicate is that a comprehension principle can be used to make explicit what is definable in a given applied language, as well as to indicate, as in logical realism, what our existential posits are regarding universals.

Thus, where L is a formal language, P^n is an n-place predicate constant not in L, and ψ is a first-order formula in which $x_1, ..., x_n$ are all of the distinct object variables occurring free, then

$$(\forall x_1)...(\forall x_n)[P(x_1, ..., x_n) \leftrightarrow \psi]$$

is said to be *a possible definition in L of P*.

Now it is just such a possible definition that is posited in ($CP!_2$). In other words, as the definiens of such a possible definition, the first-order formula ψ is implicitly understood to be a complex predicate of the language L. Of course, if the above were an explicit definition in L, then, by ($EG!_2$), the relevant instance of (CP!) follows as provable in L.

The kind of definitions that are excluded in nominalism but allowed in logical realism are the so-called **"impredicative" definitions**; that is, those that in realist terms represent properties and relations that presuppose a totality to which they belong. The definition of a least upper bound in real number theory is such a definition, for example, because, by definition, a least upper bound of a set of real numbers, is one of the upper bounds in that set.

The exclusion of impredicative definitions is also called the Poicaré-Russell **vicious circle principle**. Henri Poincaré and Bertrand Russell were the first to recognize and characterize such a principle.

4. Constructive Conceptualism

The notion of an "impredicative" definition is important in conceptualism as well as in nominalism, and it is basic to an important stage of concept-formation. Conceptualism differs from nominalism in that it assumes that there are universals, namely concepts, that are the semantic grounds for the correct application of predicate expressions. Of course, conceptualism also differs from realism in that concepts are not assumed to "exist" independently of the human capacity for thought and concept-formation.

Conceptualism is a sociobiologically based theory of the human capacity for thought and concept-formation, and, more to the point, *systematic* concept-formation. Concepts themselves are types of cognitive capacities, and it is their exercise as such that underlies the speech and mental acts that constitutes our thoughts and communications with one another.

But thought and communication exist only as coordinated activities that are systematically related to one another through the logical operations of thought.

It is with respect to the idealized closure of these logical operations that concept-formation becomes systematic. And, it is only as a result of this idealized closure that the unity of thought as a field of internal cognitive activity is possible.

Now the coordination and closure of concepts does not occur all at once in the development of human thought; nor is the structure of the closure the same at all stages in that development.

In fact, the human child proceeds through stages of cognitive development that are of increasing structural complexity, corresponding in part to the increasing complexity of the child's developing brain.

These stages, as Jean Piaget has noted, emerge as states of cognitive equilibrium with respect to certain regulatory processes that are constitutive of systematic concept-formation.

The different stages of cognitive development proceed as transformations from one state of cognitive equilibrium to another of increased structural complexity.

The need for such transformations arises from the child's interaction with the environment and the tacit realization of the inadequacy of the earlier stages to understand certain aspects of the world. The later stages are states of "increasing re-equilibration" of the intellect, in other words, so that the result is an improved representation of the world.

Now there is an important stage of cognitive equilibrium of logical operations that immediately precedes the construction of so-called "impredicative" concepts, which usually does not occur until post-adolescence.

We refer to the logic of this stage as *constructive conceptualism*. The later more mature stage at which "impredicative" concept-formation is realized is called *holistic conceptualism*, though we will generally refer to it later simply as conceptualism.

It is in constructive conceptualism that "impredicative" definitions are excluded, and this exclusion occurs in the form that the comprehension principle takes in constructive conceptualism.

This conceptualist comprehension principle, which we will call (CCP!), can be formally described as follows:

$$(\forall G_1)..(\forall G_k)(\exists F)(\forall x_1)..(\forall x_n)[F(x_1,...,x_n) \leftrightarrow \varphi],$$

where:

(1) φ is a pure second-order formula, i.e., one in which no nonlogical constants occur,

(2) F is an n-place predicate variable such that neither it nor the identity sign occur in φ,

(3) φ is "predicative" in nominalism's purely grammatical sense, i.e., no predicate variable has a bound occurrence in φ,

(4) $G_1,...,G_k$ are all of the distinct predicate variables occurring (free) in φ, and

(5) $x_1,...,x_n$ are distinct object variables.

Note: Every instance of the conceptualist principle (CCP!) is derivable from the nominalist principle (CP!). But not every instance of the nominalist principle (CP!) is derivable from an instance of the conceptualist principle (CCP!).

Now there are important reasons for each of these conditions, which we will not go into here. These reasons are based on the fact that the logic of predicate quantifiers in constructive conceptualism is free of existential presuppositions regarding predicate constants, which means that a predicate constant might not stand for a "predicative" concept unless it is either assumed or proven to do so from other assumptions—just as in free first-order logic a proper name does not denote unless it is assumed to do so.

In nominalism, however, predicate constants, as the paradigms of predication, do not differ from one another in their predicative role, which is why, under a substitutional interpretation, all predicate constants are substituends of the bound predicate variables.

On the nominalist strategy, in other words, the notion of a "predicative" context is purely grammatical in terms of logical syntax; that is, an open formula is "predicative" in nominalism just in case it contains no bound predicate variables.

In constructive conceptualism, concepts are the paradigms of predication, which means that in addition to being "predicative" in nominalism's purely grammatical sense, a predicate constant must also stand for a "predicative" concept. This is why the second-order logic of constructive conceptualism is free of existential presuppositions regarding predicate constants.

This difference in the logic of nominalism and constructive conceptualism is related to the fact that whereas "impredicative" definitions are not allowed in nominalism, they are allowed in the logic of constructive conceptualism. This is a result of the role that free predicate variables have in each of these frameworks.

In nominalism, free predicate variables must be construed as **dummy schema letters**, which in an applied language stand for arbitrary first-order formulas of that theory.

This means that *the substitution rule*,

If $\vdash \psi$, then $\vdash \psi[\varphi/G(x_1, ..., x_n)]$,

is valid on the substitutional interpretation only when φ is "predicative" in nominalism's purely grammatical sense, i.e., only when no predicate variable has a bound occurrence in φ.

The substitution rule must be restricted in this way because otherwise, by taking ψ to be the following instance of nominalism's comprehension principle, (CP!),

$$\vdash (\exists F)(\forall x_1)...(\forall x_n)[F(x_1, ..., x_n) \leftrightarrow G(x_1, ..., x_n)],$$

and substituting an arbitrary formula φ for G in this instance, regardless whether or not φ contains a bound predicate variable, then the full, unrestricted impredicative comprehension principle, (CP), would follow, when in fact (CP) is not valid in nominalism.

In other words, if the above substitution rule were valid in nominalism, then the comprehension principle (CP!) of nominalism would imply the comprehension principle of logical realism, thereby nullifying nominalism's substitutional interpretation of predicate quantifiers.

But in constructive conceptualism, the above substitution rule is valid for all formulas, regardless whether or not they contain any bound predicate variables. This is because the above instance of (CP!) is **not** also an instance of (CCP!), the comprehension principle of constructive conceptualism. This shows, incidentally, that the notion of a possible explicit definition of a predicate constant is broader in constructive conceptualism than it is in nominalism.

On the basis of the rule of substitution for all formulas, definitions in constructive conceptualism whose definiens contain a bound predicate variable will still be **noncreative** and will still allow for the **eliminability** of defined predicate constants.

Thus, even though constructive conceptualism validates only a "predicative" comprehension principle that is in accordance with the vicious circle principle, it nevertheless allows for impredicative definitions of predicate constants that do not stand for values of the bound predicate variables and that cannot therefore be existentially generalized upon.

For a more detailed account of these and other reasons for the constraints on (CCP!) see:

http://www.stoqnet.org/lat_notes.html, Course 50547: Elements of Formal Ontology, Lecture 3.

5. Ramified Constructive Conceptualism

The difference between nominalism and constructive conceptualism is similar to that between standard first-order logic and a first-order logic free of existential presuppositions regarding singular terms.

The freedom from such presuppositions for predicates in constructive conceptualism indicates how concept-formation is essentially an open process, and that part of that process is a certain tension, or disequilibrium, between the predicates and formulas that stand for concepts at a given stage of concept-formation and those that do not.

This disequilibrium in concept-formation is the real driving force of what is known as "ramified" second-order logic, though, ironically, ramified second-order logic is usually associated with nominalism and not with conceptualism.

We can close the "gap" in between predicates that stand for "predicative" concepts at a given stage of concept-formation and those that do not by introducing new predicate quantifiers in addition to the original ones.

A new comprehension principle would then be added that allowed formulas containing predicate variables bound by the original predicate quantifiers, but not also formulas containing predicate variables bound by the new predicate quantifiers.

This will close the "gap" between those formulas that stand for "predicative" concepts at the initial stage and those that do not, because the latter now stand for "predicative" concepts at the new, second stage.

But in doing this we open up a new "gap" between the formulas that stand for "predicative" concepts at the new stage and those that do not. Of course we can then go on to close this new "gap" by introducing predicate quantifiers that are new to this stage, along with a similar comprehension principle. This process is what is known as "ramification".

Formally, the process can be described in terms of a potentially infinite sequence of predicate quantifiers $\forall^1, \exists^1, ..., \forall^j, \exists^j, ...$ (for each positive integer j), all of which can be affixed to the same predicate variables. The quantifiers $(\forall^j F)$ and $(\exists^j F)$, where F is an n-place predicate variable, will then be understood to refer to all, or some, respectively, of the n-ary "'predicative"' concepts that can be formed at the *jth* stage of the potentially infinite sequence of stages of concept-formation in question.

But because open formulas representing "predicative" contexts of later stages will not be substituends of predicate variables bound by quantifiers of an earlier stage, this means that the logic of the quantifiers $\forall^j$ and $\exists^j$ must be free of existential presuppositions regarding predicate expressions, which is why the comprehension principle for this logic must be closed with respect to all the predicate variables occurring free in the comprehending formula.

Thus, as applied at the *jth* stage, *the ramified conceptualist comprehension principle*, (RCCP!), that is validated in this framework is the following:

(RCCP!) $(\forall^j G_1)...(\forall^j G_k)(\exists^j F)(\forall x_1)...(\forall x_n)[F(x_1, ..., x_n)$

$$\leftrightarrow \varphi],$$

where

(1) $G_1, ..., G_k$ are all of the predicate variables occurring free in φ;

(2) F is an n-place predicate variable not occurring free in φ;

(3) $x_1, ..., x_n$ are distinct objectual variables; and

(4) φ is a formula in which no nonlogical constants occur and in which (a) the identity sign also does not occur, and (b) in which no predicate variable is bound by a quantifier of a stage $> j$, i.e., for all $i \geq j$, neither $\forall^j$ nor $\exists^j$ occurs in φ.

The process of concept-formation that we are describing here amounts to a type of *reflective abstraction* that involves a projection of previously constructed concepts onto a new plane of thought where they are reorganized under the closure conditions of new laws of concept-formation characteristic of the new stage in question.

This pattern of reflective abstraction is precisely what is represented by the ramified comprehension principle (RCCP!) and the logic of constructive conceptualism.

Each successive stage of concept-formation in the ramified hierarchy is generated by a disequilibrium, or conceptual tension, between the formulas that stand for the "predicative" concepts of the preceding stage, as opposed to those that do not. Thus, in order to close the "gap" between formulas that stand for "predicative" concepts and those do not, we must proceed through a potentially infinite sequence of stages of concept-formation.

What moves us on from one stage of concept-formation to the next in constructive conceptualism is a drive for closure, where all predicates stand for concepts.

Such a closure cannot be realized in constructive conceptualism, of course, where the principal constraint guiding the formation of "predicative" concepts is their being specifiable by conditions that are in accord with the so-called vicious circle principle.

But the particular pattern of reflective abstraction that corresponds to this constraint is not all there is to concept-formation, and, in fact, as a pattern that represents a drive for closure, it contains the seeds of its own transcendence to a new plane of thought where such closure is achieved.

6. Holistic Conceptualism

As the history of mathematics has shown, concept-formation is not constrained by the vicious circle principle. This is because after reaching what Piaget calls the stage of **formal operational thought**, certain new patterns of concept-formation are realizable, albeit usually only in post-adolescence.

One such pattern involves an idealized transition to a limit, where "impredicative" concept-formation becomes possible, i.e., where the restrictions imposed by the vicious circle principle are transcended.

In the case of our conceptualist ramified logic, the idealized transition to a limit is conceptually similar to, but ontologically different from, an actual transition to a limit at an infinite stage of concept-formation — that is, a stage of concept-formation that is not only the summation of all of the finite stages of the ramified hierarchy but also one that is closed with respect to itself.

Ontologically, of course, there cannot be an infinite stage of concept-formation, but that is not to say that an idealized transition to a limit is conceptually impossible as well.

In fact, such an idealized transition to a limit is precisely what is assumed to be possible in *holistic conceptualism*, and it is assumed to be possible, moreover, on the basis of the pattern of reflective abstraction represented by the ramified comprehension principle (RCCP!).

That is, in holistic conceptualism, the drive for closure upon which the pattern of reflective abstraction of ramified constructive conceptualism is based is finally achieved, albeit only as the result of **an idealized transition to a limit** and not on the basis of an actual transition.

In this way, by means of a mechanism of autoregulation that enables us to construct stronger and more complex logical systems out of weaker ones, conceptualism is able to validate not only the ramified conceptualist comprehension principle but also the full, unqualified "impredicative" comprehension principle **(CP)** of "standard" second-order logic.

There is no comparable mechanism in nominalism, it is important to note, that can similarly lead to a validation of the impredicative comprehension principle (CP).

What is inadequate about nominalism and the logic of constructive conceptualism is that neither alone can provide an account of the kind of impredicative concept-formation that is necessary for the development and use of the theory of real numbers, and which, as a matter of cultural history, we have in fact achieved since the nineteenth century.

The concept of a least upper bound, for example, or of the limit of a converging sequence of rational numbers, is an impredicative concept that was not acquired by the mathematical community until a century and a half ago.

And although the concept of a least upper bound is now not usually a part of a person's conceptual repertoire until post-adolescence, nevertheless, with proper training and conceptual development, it is a concept that most of us can come to acquire as a cognitive capacity.

Yet, notwithstanding these facts of cultural history and conceptual development, it is also a concept that cannot be accounted for from within the framework of either nominalism or constructive conceptualism.

The constraints of the vicious circle principle, at least in the way they apply to concept-formation, simply do not conform to the facts of conceptual development in an age of advanced scientific knowledge.

Note, however, that the validation of the full comprehension principle in holistic conceptualism, which we will hereafter refer to simply as conceptualism, does not mean that the logic of constructive conceptualism is no longer a useful part of conceptualism.

What it does mean is that although all predicates stand for "predicable" concepts, not all predicates stand for "predicative" concepts, and in order to represent such a situation we need the logic of constructive conceptualism as a proper part of holistic conceptualism.

In closing this lecture, let us raise the question of how, if at all, we are to distinguish holistic conceptualism from logical realism if the full impredicative comprehension principle (CP) is valid in both frameworks. Is there no difference then between holistic conceptualism and logical realism?