Links. Relating different physical systems through the common QFT algebraic structure

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Summary. In this report I review some aspects of the algebraic structure of QFT related with the doubling of the degrees of freedom of the system under study. I show how such a doubling is related to the characterizing feature of QFT consisting in the existence of infinitely many unitarily inequivalent representations of the canonical (anti-)commutation relations and how this is described by the q-deformed Hopf algebra. I consider several examples, such as the damped harmonic oscillator, the quantum Brownian motion, thermal field theories, squeezed states, classical-to-quantum relation, and show the analogies, or links, among them arising from the common algebraic structure of the q-deformed Hopf algebra.

1 Introduction

Since several years I am pursuing the study of the vacuum structure in quantum field theory (QFT) through a number of physical problems such as boson condensation and the infrared effects in spontaneously broken symmetry gauge theories, coherent domain formation and defect formation, soliton solutions, particle mixing and oscillation, the canonical formalism for quantum dissipation and unstable states, the quantization in curved background, thermal field theories, quantum-to-classical relationship. In this paper I would like to share with the reader the satisfying feeling of a unified view of several distinct physical phenomena emerging from such a study of the QFT vacuum structure. Besides such a pleasant feeling, there is a concrete interest in pointing out the analogies ("links") among these phenomena, which arises since these links provide a great help not only in the formulation of their mathematical description, but also in the understanding of the physics involved in them. Such a 'compared study' also reflects back to a deeper understanding of structural aspects of the same QFT formalism.

Quite often QFT is presented as an extension of quantum mechanics (QM) to the relativistic domain. Sometimes it is referred to as "second quantization". Of course, the reasons for that come from the historical developments in the formulation of the quantum theory of elementary particle physics and solid state physics.

However, a closer view to the formalism of QFT shows that it is not necessarily related with the relativistic domain and it is not simply a "second" quantization recipe subsequent the quantization procedure in QM. For example, the QFT formalism is widely used, with great success, in condensed matter physics, e.g. in the formulation of superconductivity, of ferromagnetism, etc., where typically one does not refer to the relativistic domain. On the other hand, in dealing with fermion fields one cannot rely on the quantization scheme adopted in QM for boson creation and annihilation operators.

As it will appear in the following, QFT is drastically different from QM. The main reason for this resides in the fact that the well known von Neumann theorem, which characterizes in a crucial way the structure of QM [1, 2], does not hold in QFT. In QM the von Neumann theorem states that for systems with a finite number of degrees of freedom all the representations of the canonical commutation relations (ccr) are unitarily equivalent. This means that they are physically equivalent; namely, the representations of the ccr are related by unitary operators and, as well known, physical observables are invariant under the action of unitary operators. Their value is therefore the same independently of the representation one choses to work in. Such a choice is thus completely arbitrary and does not affect the physics one is going to describe. The situation is quite different in QFT where the von Neumann theorem does not hold. Indeed, the hypothesis of finite number of degrees of freedom on which the theorem rests is not satisfied since fields involve by definition infinitely many degrees of freedom. As a consequence, infinitely many unitarily inequivalent (ui) representations of the ccr are allowed to exist [3, 4, 5]. The existence of ui representations is thus a characterizing feature of QFT and a full series of physically relevant consequences follows.

One of the aspects I will discuss below is related with the algebraic structure of QFT. I will show that the relevant algebra underlying the QFT formalism is the Hopf algebra, and this underlies the existence of the ui representations. It manifests in the doubling of the system degrees of freedom and its q-deformation bears deep physical meaning. In the first part of the paper, I will start by considering some aspects of the two-slit experiment. This is a typical subject in QM where quantum features fully show up. The discussion turns out to be useful for the subsequent discussion of the q–deformed Hopf algebra structure of QFT [6, 7] and it also provides a good example where the quantum-to-classical relation manifests itself.

The q-deformation of the Hopf algebra will be shown to be also related with quantum dissipation and with thermal field theory, where the description of statistical thermal averages of observables in operatorial terms is made possible by exploiting the existence of infinitely many ui representations [5, 8, 9, 10]. Recognizing that a symplectic structure with classical dynamics is embedded in the space of the ui representations of ccr in QFT [10] leads to show that trajectories (i.e. a sequence of phase transitions) in such a space may satisfy, under convenient conditions, the criteria for chaoticity prescribed by nonlinear classical dynamics. In a figurate way one could say that a *classical blanket* covers the space of the QFT ui representations. Moving on such a blanket describes (phase) transitions among the representations.

The problem of the interplay between 'classical and quantum' is indeed another topic on which I will comment on in this paper and I will show that it is intrinsic to the mathematical structure of QFT [10, 11]. The phenomenon of decoherence in QM and the related emergence of classicality from the quantum realm is analyzed in detail in the literature [12]. Similarly, although based on different formal and conceptual frame, the emergence of macroscopic ordered patterns and classically behaving structures out of a QFT (not QM!) dynamics via the spontaneous breakdown of symmetry is since long well known [5, 13]. Examples of such classically behaving *macroscopic quantum systems* are crystals, ferromagnets, superconductors, superfluids. These are quantum systems not in the trivial sense that they, as all other systems, are made of quantum components, but in the sense that their macroscopic behavior, characterized by the classical (c-number) observable called order parameter, cannot be explained without recourse to the underlying quantum field dynamics.

On the other hand, in recent years the problem of quantization of a classical theory has attracted much attention in gravitation theories and in non-hamiltonian dissipative system theories, where a novel perspective has been proposed [14] according to which the 'emergence' of the quantum-like behavior from a classical frame may occur. I will comment in particular on classical deterministic systems with dissipation (information loss) which are found to exhibit quantum behavior under convenient conditions [14, 15, 16]. The paper is organized as follows: the doubling the degrees of freedom is discussed in Sec. 2, the two-slit experiment in Sec. 2.1, unitarily inequivalent representations in QFT in Sec. 3, quantum dissipation in Sec. 3.1, the thermal connection and the arrow of time in Sec. 3.2, two-mode squeezed coherent states in Sec. 4, quantum Brownian motion in Sec. 5, the dissipative noncommutative plane in Sec. 6. Thermal field theory in the operatorial formalism (TFD) is presented in Sec. 7. In section 8 the q–deformed Hopf algebra is shown to be a basic feature of QFT. Entropy as a measure of entanglement and the trajectories in the space of the ui representations are discussed in Sec. 9 and 10 respectively. Deterministic dissipative systems are considered in Sec. 11 with respect to the quantization problem. Section 12 is devoted to conclusions. In this paper I have not considered the doubling of the degrees of freedom in inflationary models and in the problem of the quantization of the matter field in a curved background. The interest reader is referred to the papers [17, 18, 19].

2 Doubling the degrees of freedom

One of the main features underlying the QFT formalism is the doubling of the degrees of freedom of the system under study. Such a doubling is not simply a mathematical tool useful to describe our system. On the contrary, it bears a physical meaning. It also appears to be an essential feature of QM, as I will show in the examples I am going to discuss in this paper.

The standard formalism of the density matrix [20, 21] and of the associated Wigner function [22] suggests tha one may describe a quantum particle by splitting the single coordinate $x(t)$ into two coordinates $x_+(t)$ (going forward in time) and $x_-(t)$ (going backward in time). Indeed, the standard expression for the Wigner function is [22],

$$W(p,x,t) = \frac{1}{2\pi\hbar} \int \psi^* \left(x - \frac{1}{2}y, t \right) \psi \left(x + \frac{1}{2}y, t \right) e^{-i\frac{py}{\hbar}} dy \ , \tag{1}$$

where

$$x_\pm = x \pm \frac{1}{2} y \ . \tag{2}$$

By employing the Schwinger quantum operator action principle, or recalling the mean value of a quantum operator

$$\bar{A}(t) = (\psi(t)|A|\psi(t)) = \tag{3}$$

$$\int\int \psi^*(x_-,t)\,(x_-|A|x_+)\,\psi(x_+,t)dx_+dx_- = \tag{4}$$

$$\int\int (x_+|\rho(t)|x_-)(x_-|A|x_+)dx_+dx_-. \tag{5}$$

one requires the density matrix

$$W(x,y,t) = (x_+|\rho(t)|x_-) = \psi^*(x_-,t)\psi(x_+,t)\ , \tag{6}$$

to follow two copies of the Schrödinger equation: the forward in time motion and the backward in time motion, respectively. These motions are controlled by the two Hamiltonian operators $H_\pm$:

$$i\hbar\frac{\partial\psi(x_\pm,t)}{\partial t} = H_\pm\psi(x_\pm,t), \tag{7}$$

which gives

$$i\hbar\frac{\partial(x_+|\rho(t)|x_-)}{\partial t} = \mathcal{H}\ (x_+|\rho(t)|x_-), \tag{8}$$

where

$$\mathcal{H} = H_+ - H_-. \tag{9}$$

Using two copies of the Hamiltonian (i.e. $H_\pm$) operating on the outer product of two Hilbert spaces $\mathcal{F}_+ \otimes \mathcal{F}_-$ has been implicitly required in QM since the very beginning of the theory. For example, from Eqs.(8), (9) one finds immediately that the eigenvalues of $\mathcal{H}$ are directly the Bohr transition frequencies $\hbar\omega_{nm} = E_n - E_m$ which was the first clue to the explanation of spectroscopic structure.

The notion that a quantum particle has two coordinates $x_\pm(t)$ moving at the same time is therefore central [23].

In conclusion, the density matrix and the Wigner function *require* the introduction of a "doubled" set of coordinates, $(x_\pm, p_\pm)$ (or (x, p_x) and (y, p_y)).

Let me show how the doubling of the coordinates works in the remarkable example of the two-slit diffraction experiment. Here I will shortly summarize the discussion reported in [24].

2.1 The two-slit experiment

In order to derive the diffraction pattern it is required to know the wave function $\psi_0(x)$ of the particle when it "passes through the slits" at time zero. In other words, one searches for the density matrix

$$(x_+|\rho_0|x_-) = \psi_0^*(x_-)\psi_0(x_+). \tag{10}$$

The probability density for the electron to be found at position x at the detector screen at a later time t is written as

$$P(x,t) = (x|\rho(t)|x) = \psi^*(x,t)\psi(x,t) \tag{11}$$

in terms of the solution $\psi(x,t)$ to the free particle Schrödinger equation

$$\psi(x,t) = \left(\frac{M}{2\pi\hbar it}\right)^{1/2} \int_{-\infty}^{\infty} e^{[\frac{i}{\hbar}A(x-x',t)]}\psi_0(x')dx', \tag{12}$$

where

$$A(x-x',t) = \frac{M(x-x')^2}{2t} \tag{13}$$

is the Hamilton-Jacobi action for a classical free particle to move from x' to x in a time t. Eqs. (10)-(13) then imply that

$$P(x,t) = \frac{M}{2\pi\hbar t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\left[iM\frac{(x-x_+)^2-(x-x_-)^2}{2\hbar t}\right]}(x_+|\rho_0|x_-)dx_+dx_-. \tag{14}$$

Eq.(14) shows that $P(x,t)$ would not oscillate in x, i.e. there would not be the usual quantum diffraction, if $x_+ = x_-$. In Eq.(14), in order to have quantum interference the forward in time action $A(x-x_+,t)$ must be different from the backward in time action $A(x-x_-,t)$: the non-trivial dependence of the density matrix $(x_+|\rho_0|x_-)$ when the electron "passes through the slits" on the difference $(x_+ - x_-)$ crucially determines the quantum nature of the phenomenon.

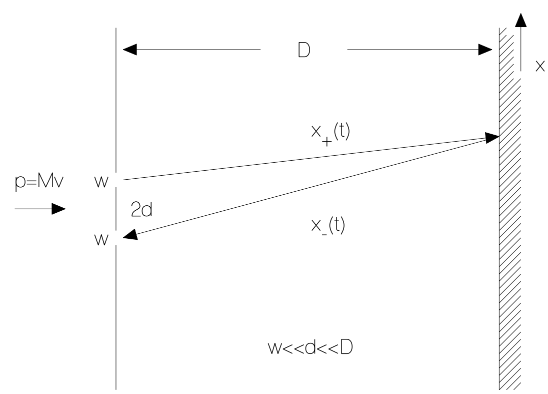

Fig. 1. Two slit experiment.

In the quantum diffraction experiment the experimental apparatus is prepared so that $w \ll d \ll D$, with w the opening of the slits which are separated by a distance $2d$. D is the distance between the slits and the screen (Fig.1). The diffraction pattern is described by $|x| \gg |x_\pm|$. By defining $K = \frac{Mvd}{\hbar D}$, $\beta = \frac{w}{d}$, with $v = D/t$ the velocity of the incident electron, Eq.(14) leads [24] to the usual result

$$P(x,D) \approx \frac{4}{\pi\beta Kx^2}\ \cos^2(Kx)\sin^2(\beta Kx), \tag{15}$$

where the initial wave function

$$\psi_0(x) = \frac{1}{\sqrt{2}}\Big[\phi(x-d) + \phi(x+d)\Big], \tag{16}$$

with $\phi(x) = \frac{1}{\sqrt{w}}$ if $|x| \leq \frac{w}{2}$ and zero otherwise, has been used. From Eqs.(10) and (16) we have

$$\begin{aligned}(x_+|\rho_0|x_-) = \; & \frac{1}{2}\Big\{\phi(x_+ - d)\phi(x_- - d) + \phi(x_+ + d)\phi(x_- + d) \\ & +\phi(x_+ - d)\phi(x_- + d) + \phi(x_+ + d)\phi(x_- - d)\Big\}. \end{aligned} \tag{17}$$

In the rhs of Eq.(17) the first and the second terms describe the classical processes of the particle going forward and backward in time through slit 1 and going forward and backward in time through slit 2, respectively. In these processes it is $x_+(t) = x_-(t)$ and in such cases no diffraction is observed on the screen. The third term and the fourth term describe the particle going forward in time through slit 1 and backward in time through slit 2, or forward in time through slit 2 and backward in time through slit 1, respectively. These are the terms generating quantum interference since $|x_+(t) - x_-(t)| > 0$.

In conclusion, the doubling the system coordinates, $x(t) \rightarrow (x_+(t), x_-(t))$ plays a crucial rôle in the description of the quantum system. If $x(t) \equiv x_+(t) \equiv x_-(t)$, then the system behavior appears to be a classical one. When forward in time and backward in time motions are (at the same time) unequal $x_+(t) \neq x_-(t)$, then the system is behaving in a quantum mechanical fashion and interference patterns appear in measured position probability densities.

I will not comment further on the two-slit experiment. In the following Section I go back to the general discussion of the doubling of the degrees of freedom and of its meaning in QFT.

3 Unitarily inequivalent representations in QFT

The mathematical rôle and the physical meaning of the doubling of the degrees of freedom fully appears in dealing with phase transitions, with equilibrium and non-equilibrium thermal field theories and with dissipative, open systems. In these cases the doubling of the degrees of freedom appears to be a structural feature of QFT since it strictly relates with the existence of the unitarily inequivalent representations of the ccr in QFT.

Let me consider the case of dissipation [25, 26, 27]. I will discuss the canonical quantization of the damped (simple) harmonic oscillator (dho), which is a simple prototype of dissipative system.

3.1 Quantum dissipation

Dissipation enters into our considerations if there is a coupling to a thermal reservoir yielding a mechanical resistance R. According to the discussion in Section 2, the equation of motion for the density matrix is given by Eq. (8), where now the Hamiltonian $\mathcal{H}$ for motion in the (x_+, x_-) plane is [23, 24]

$$\mathcal{H} = \frac{1}{2M}(p_+ - \frac{R}{2}\,x_-)^2 - \frac{1}{2M}(p_- + \frac{R}{2}\,x_+)^2 + U(x_+) - U(x_-)\;, \tag{18}$$

where $p_{\pm} = -i\hbar \frac{\partial}{\partial x_{\pm}}$. In order to simplify the discussion, it is convenient, without loss of generality, to make an explicit (simple) choice for $U(x_{\pm})$, say $U(x_{\pm}) = \frac{1}{2}\kappa x_{\pm}^2$. By choosing as doubled coordinates the pair (x, y) with

$$y = x_+ - x_- \ , \tag{19}$$

the Hamiltonian (18) can be derived from the Lagrangian (see [25] - [29])

$$L = M\dot{x}\dot{y} + \frac{1}{2}R(x\dot{y} - \dot{x}y) - \kappa xy \ . \tag{20}$$

The system described by (20) is sometimes called Bateman's dual system [29]. I observe that the doubling imposed by the density matrix and the Wigner function formalism, as seen in Section 2, here finds its physical justification in the fact that the canonical quantization scheme can only deal with an isolated system. In the present case our system has been assumed to be coupled with a thermal reservoir and it is then necessary to *close* the system by including the reservoir. This is achieved by doubling the phase-space dimensions [25, 26]. Eq. (20) is indeed the closed system Lagrangian.

By varying Eq. (20) with respect to y gives

$$M\ddot{x} + R\dot{x} + \kappa x = 0 \quad , \tag{21}$$

whereas variation with respect to x gives

$$M\ddot{y} - R\dot{y} + \kappa y = 0 \quad , \tag{22}$$

which is the *time reversed* ($R \to -R$) of Eq. (21). The physical meaning of the doubled degree of freedom y is now manifest: y may be thought of as describing an effective degree of freedom for the reservoir to which the system (21) is coupled. The canonical momenta are given by $p_x \equiv \frac{\partial L}{\partial \dot{x}} = M\dot{y} - \frac{1}{2}Ry$; $p_y \equiv \frac{\partial L}{\partial \dot{y}} = M\dot{x} + \frac{1}{2}Rx$. For a discussion of Hamiltonian systems of this kind see also [30, 31]. Canonical quantization is performed by introducing the commutators

$$[x, p_x] = i\,\hbar = [y, p_y], \quad [x, y] = 0 = [p_x, p_y] \ , \tag{23}$$

and the corresponding sets of annihilation and creation operators

$$\alpha \equiv \left(\frac{1}{2\hbar\Omega}\right)^{\frac{1}{2}} \left(\frac{p_x}{\sqrt{M}} - i\sqrt{M}\Omega x\right), \quad \alpha^\dagger \equiv \left(\frac{1}{2\hbar\Omega}\right)^{\frac{1}{2}} \left(\frac{p_x}{\sqrt{M}} + i\sqrt{M}\Omega x\right), \tag{24}$$

$$\beta \equiv \left(\frac{1}{2\hbar\Omega}\right)^{\frac{1}{2}} \left(\frac{p_y}{\sqrt{M}} - i\sqrt{M}\Omega y\right), \quad \beta^\dagger \equiv \left(\frac{1}{2\hbar\Omega}\right)^{\frac{1}{2}} \left(\frac{p_y}{\sqrt{M}} + i\sqrt{M}\Omega y\right), \tag{25}$$

$$[\,\alpha, \alpha^\dagger\,] = 1 = [\,\beta, \beta^\dagger\,] \quad , \quad [\,\alpha, \beta\,] = 0 = [\,\alpha, \beta^\dagger\,] \quad . \tag{26}$$

I have introduced $\Omega \equiv \left[\frac{1}{M}\left(\kappa - \frac{R^2}{4M}\right)\right]^{\frac{1}{2}}$, the common frequency of the two oscillators Eq. (21) and Eq. (22), assuming Ω real, hence $\kappa > \frac{R^2}{4M}$ (case of no overdamping).

In Section 5 I show that, at quantum level, the β modes allow quantum noise effects arising from the imaginary part of the action [23]. Moreover, in Section 8 the modes α and β will be shown to be the modes involved in the coproduct operator of the underlying q–deformed Hopf algebra structure. The q-deformation parameter turns out to be a function of R, M and t.

By using the canonical linear transformations $A \equiv \frac{1}{\sqrt{2}}(\alpha + \beta)$, $B \equiv \frac{1}{\sqrt{2}}(\alpha - \beta)$, the quantum Hamiltonian H is then obtained [25, 26] as

$$H = H_0 + H_I \quad , \tag{27}$$

$$H_0 = \hbar \Omega (A^\dagger A - B^\dagger B) \quad , \quad H_I = i\hbar \Gamma (A^\dagger B^\dagger - AB) \quad , \tag{28}$$

where the decay constant for the classical variable $x(t)$ is denoted by $\Gamma \equiv \frac{R}{2M}$.

In conclusion, *the states generated by* $B^\dagger$ *represent the sink where the energy dissipated by the quantum damped oscillator flows: the* B*-oscillator represents the reservoir or heat bath coupled to the* A*-oscillator.*

The dynamical group structure associated with the system of coupled quantum oscillators is that of $SU(1,1)$. The two mode realization of the algebra $su(1,1)$ is indeed generated by $J_+ = A^\dagger B^\dagger$, $J_- = J_+^\dagger = AB$, $J_3 = \frac{1}{2}(A^\dagger A + B^\dagger B + 1)$, $[\, J_+, J_- \,] = -2J_3$, $[\, J_3, J_\pm \,] = \pm J_\pm$. The Casimir operator $\mathcal{C}$ is $\mathcal{C}^2 \equiv \frac{1}{4} + J_3^2 - \frac{1}{2}(J_+ J- + J_- J_+) = \frac{1}{4}(A^\dagger A - B^\dagger B)^2$.

I also observe that $[\, H_0, H_I \,] = 0$. The time evolution of the vacuum $|0> \equiv |n_A = 0, n_B = 0> = |0> \otimes |0>$, $(A \otimes 1)|0> \otimes |0> \equiv A|0> = 0$; $(1 \otimes B)|0> \otimes |0> \equiv B|0> = 0$, is controlled by H_I

$$|0(t)> = \exp\left(-it\frac{H}{\hbar}\right)|0> = \exp\left(-it\frac{H_I}{\hbar}\right)|0>$$

$$= \frac{1}{\cosh(\Gamma t)} \exp\left(\tanh(\Gamma t) A^\dagger B^\dagger\right)|0> \quad , \tag{29}$$

$$< 0(t)|0(t) > = 1 \qquad \forall t \ , \tag{30}$$

$$\lim_{t\to\infty} < 0(t)|0> \propto \lim_{t\to\infty} \exp(-t\Gamma) = 0 \quad . \tag{31}$$

Notice that once one sets the initial condition of positiveness for the eigenvalues of H_0, such a condition is preserved by the time evolution since H_0 is the Casimir operator (it commutes with H_I). In other words, there is no danger of dealing with energy spectrum unbounded from below. Time evolution for creation and annihilation operators is given by

$$A \mapsto A(t) = \mathrm{e}^{-i\frac{t}{\hbar}H_I} A \ \mathrm{e}^{i\frac{t}{\hbar}H_I} = A\cosh(\Gamma t) - B^\dagger \sinh(\Gamma t) \ , \tag{32}$$

$$B \mapsto B(t) = \mathrm{e}^{-i\frac{t}{\hbar}H_I} B \ \mathrm{e}^{i\frac{t}{\hbar}H_I} = B\cosh(\Gamma t) - A^\dagger \sinh(\Gamma t) \tag{33}$$

and h.c.. I note that Eqs. (32) and (33) are Bogolubov transformations: they are canonical transformations preserving the ccr. Eq. (31) expresses the instability (decay) of the vacuum under the evolution operator $\exp\left(-it\frac{H_I}{\hbar}\right)$. In other words, time evolution leads out of the Hilbert space of the states. *This means that the QM framework is not suitable for the canonical quantization of the damped harmonic oscillator.* A way out from such a difficulty is provided by QFT [25]: the proper way to perform the canonical quantization of the dho turns out to be working in the framework of QFT. In fact, for many degrees of freedom the time evolution operator $\mathcal{U}(t)$ and the vacuum are formally (at finite volume) given by

$$\mathcal{U}(t) = \prod_{\kappa} \exp\Big(\Gamma_{\kappa} t \big(A_{\kappa}^{\dagger} B_{\kappa}^{\dagger} - A_{\kappa} B_{\kappa}\big)\Big), \tag{34}$$

$$|0(t)> = \prod_{\kappa} \frac{1}{\cosh(\Gamma_{\kappa} t)} \exp\big(\tanh(\Gamma_{\kappa} t) A_{\kappa}^{\dagger} B_{\kappa}^{\dagger}\big)|0> \quad , \tag{35}$$

with $<0(t)|0(t)>= 1,\ \forall t$. Using the continuous limit relation $\sum_{\kappa} \mapsto \frac{V}{(2\pi)^3}\int d^3\kappa$, in the infinite-volume limit we have (for $\int d^3\kappa\ \Gamma_{\kappa}$ finite and positive)

$$<0(t)|0> \rightarrow 0 \ \text{ as } \ V \rightarrow \infty \quad \forall\ t\ , \tag{36}$$

and in general, $<0(t)|0(t')> \rightarrow 0$ as $V \rightarrow \infty \ \ \forall\ t$ and $t',\ t' \neq t$. At each time t a representation $\{|0(t)>\}$ of the ccr is defined and turns out to be ui to any other representation $\{|0(t')>,\ \forall t' \neq t\}$ in the infinite volume limit. In such a way the quantum dho evolves in time through ui representations of ccr (*tunneling*). I remark that $|0(t)>$ is a two-mode time dependent generalized coherent state [32, 33]. Also note that

$$\mathcal{N}_{A_{\kappa}}(t) = <0(t)|A_{\kappa}^{\dagger} A_{\kappa}|0(t)> = \sinh^2 \Gamma t\ , \tag{37}$$

The Bogolubov transformations, Eqs. (32) and (33) can be implemented for every κ as inner automorphism for the algebra $su(1,1)_{\kappa}$. At each time t one has a copy $\{A_{\kappa}(t), A_{\kappa}^{\dagger}(t), B_{\kappa}(t), B_{\kappa}^{\dagger}(t)\ ;\ |0(t)>\ |\forall\kappa\}$ of the original algebra induced by the time evolution operator which can thus be thought of as a generator of the group of automorphisms of $\bigoplus_{\kappa} su(1,1)_{\kappa}$ parameterized by time t (we have a realization of the operator algebra at each time t, which can be implemented by Gel'fand-Naimark-Segal construction in the C*-algebra formalism [3, 34]). Notice that the various copies become unitarily inequivalent in the infinite-volume limit, as shown by Eqs. (36): the space of the states splits into ui representations of the ccr each one labeled by time parameter t. As usual, one works at finite volume and only at the end of the computations the limit $V \rightarrow \infty$ is performed.

Finally, I note that the "negative" kinematic term in the Hamiltonian (28) (or (18)) also appears in two-dimensional gravity models where, in general, two different strategies are adopted in the quantization procedure [35]: the Schrödinger representation approach, where no negative norm appears, and the string/conformal field theory approach where negative norm states arise as in Gupta-Bleurer electrodynamics.

3.2 The thermal connection and the arrow of time

It is useful [25] to introduce the functional $\mathcal{F}_A$ for the A-modes

$$\mathcal{F}_A \equiv <0(t)|\Big(H_A - \frac{1}{\beta} S_A\Big)|0(t)> \quad , \tag{38}$$

where β is a non-zero c-number, H_A is the part of H_0 relative to A- modes only, namely $H_A = \sum_{\kappa} \hbar\Omega_{\kappa} A_{\kappa}^{\dagger} A_{\kappa}$, and the S_A is given by

$$S_A \equiv -\sum_{\kappa}\Big\{A_{\kappa}^{\dagger} A_{\kappa} \ln\sinh^2\big(\Gamma_{\kappa} t\big) - A_{\kappa} A_{\kappa}^{\dagger} \ln\cosh^2\big(\Gamma_{\kappa} t\big)\Big\} \quad . \tag{39}$$

One then considers the extremal condition $\frac{\partial \mathcal{F}_A}{\partial \vartheta_{\kappa}} = 0 \quad \forall\kappa\ ,\ \vartheta_{\kappa} \equiv \Gamma_{\kappa} t$ to be satisfied in each representation, and using the definition $E_{\kappa} \equiv \hbar\Omega_{\kappa}$, one finds

$$\mathcal{N}_{A_\kappa}(t) = \sinh^2\big(\Gamma_\kappa t\big) = \frac{1}{e^{\beta(t)E_\kappa} - 1} \quad , \tag{40}$$

which is the Bose distribution for A_κ at time t, *provided* $\beta(t)$ is the (time-dependent) inverse temperature. Inspection of Eqs. (38) and (39) then suggests that $\mathcal{F}_A$ and S_A can be interpreted as the *free energy* and the *entropy*, respectively. I will comment more about this in Section 7 and 9.

$\{|0(t)>\}$ is thus recognized to be a representation of the ccr at finite temperature (it turns out to be equivalent to the thermo field dynamics (TFD) representation $\{|0(\beta)>\}$ [5, 8], see Section 7). Use of Eq. (39) shows that

$$\frac{\partial}{\partial t}|0(t)> = -\left(\frac{1}{2}\frac{\partial \mathcal{S}}{\partial t}\right)|0(t)> \quad . \tag{41}$$

One thus see that $i\left(\frac{1}{2}\hbar\frac{\partial \mathcal{S}}{\partial t}\right)$ is the generator of time translations, namely time evolution is controlled by the entropy variations [36]. It is remarkable that the same dynamical variable $\mathcal{S}$ whose expectation value is formally the entropy also controls time evolution: damping (or, more generally, dissipation) implies indeed the choice of a privileged direction in time evolution (*arrow of time*) with a consequent breaking of time-reversal invariance.

One may also show that $d\mathcal{F}_A = dE_A - \frac{1}{\beta}d\mathcal{S}_A = 0$, which expresses the first principle of thermodynamics for a system coupled with environment at constant temperature and in absence of mechanical work. As usual, one may define heat as $dQ = \frac{1}{\beta}dS$ and see that the change in time $d\mathcal{N}_A$ of particles condensed in the vacuum turns out into heat dissipation dQ:

$$dE_A = \sum_\kappa \hbar\Omega_\kappa \dot{\mathcal{N}}_{A_\kappa}(t)dt = \frac{1}{\beta}dS = dQ \ . \tag{42}$$

Here $\dot{\mathcal{N}}_{A_\kappa}$ denotes the time derivative of $\mathcal{N}_{A_\kappa}$.

It is interesting to observe that the thermodynamic arrow of time, whose direction is defined by the increasing entropy direction, points in the same direction of the cosmological arrow of time, namely the inflating time direction for the expanding Universe. This can be shown by considering indeed the quantization of inflationary models [17] (see also [18]). The concordance between the two arrows of time (and also with the psychological arrow of time, see refs. [37]) is not at all granted and is a subject of an ongoing debate (see, e.g., [38]).

In Section 6 I will show that quantum dissipation induces a dissipative phase interference [24], analogous to the Aharonov-Bohm phase [39], and a noncommutative geometry in the plane (x_+, x_-) [40].

The quantum dissipation Lagrangian model discussed above is strictly related with the squeezed coherent states in quantum optics and with the quantum Brownian motion. I will briefly discuss these two topics in follolwing Sections.

4 Two-mode squeezed coherent states

Here I will only mention that in the quantum damped oscillator treatment presented above the time evolution operator $\mathcal{U}(t)$ written in terms of the α and β modes (Eqs. (24) and (25)) is given by

$$\mathcal{U}(t) \equiv \exp\left(-it\frac{H_I}{\hbar}\right) = \prod_\kappa \exp\Big(-\frac{\theta_\kappa}{2}\big(\alpha_\kappa^2 - \alpha_\kappa^{\dagger 2}\big)\Big) \exp\Big(\frac{\theta_\kappa}{2}\big(\beta_\kappa^2 - \beta_\kappa^{\dagger 2}\big)\Big)$$
$$\equiv \prod_\kappa \hat{S}_\alpha(\theta_\kappa)\hat{S}_\beta(-\theta_\kappa)\ , \qquad (43)$$

with $\hat{S_\alpha}(\theta_\kappa) \equiv \exp\big(-\frac{\theta_\kappa}{2}\big({\alpha_\kappa}^2 - {\alpha_\kappa}^{\dagger 2}\big)\big)$ and similar expression for $\hat{S_\beta}(-\theta_\kappa)$ with β and $\beta^\dagger$ replacing α and $\alpha^\dagger$, respectively. The operators $\hat{S}_\alpha(\theta_\kappa)$ and $\hat{S}_\beta(-\theta_\kappa)$ are the squeezing operators for the α_κ and the β_κ modes, respectively, as well known in quantum optics [41]. The set $\theta \equiv \{\theta_\kappa \equiv \Gamma_\kappa t\}$ as well as each θ_κ for all κ is called the squeezing parameter. The state $|0(t)>$ is thus a squeezed coherent states at each time t.

To illustrate the effect of the squeezing, let me focus the attention only on the α_κ modes for sake of definiteness. For the β modes one can proceed in a similar way. As usual, for given κ I express the α mode in terms of conjugate variables of the corresponding oscillator. By using dimensionless quantities I thus write $\alpha = X + iY$, with $[X, Y] = \frac{i}{2}$. The uncertainty relation is $\Delta X \Delta Y = \frac{1}{4}$, with $\Delta X^2 = \Delta Y^2 = \frac{1}{4}$ for (minimum uncertainty) coherent states. The squeezing occurs when $\Delta X^2 < \frac{1}{4}$ *and* $\Delta Y^2 > \frac{1}{4}$ (or $\Delta X^2 > \frac{1}{4}$ *and* $\Delta Y^2 < \frac{1}{4}$) in such a way that the uncertainty relation remains unchanged. Under the action of $\mathcal{U}(t)$ the variances ΔX and ΔY are indeed squeezed as

$$\Delta X^2(\theta) = \Delta X^2 \exp(2\theta)\ \ , \qquad \Delta Y^2(\theta) = \Delta Y^2 \exp(-2\theta)\ \ . \qquad (44)$$

For the tilde-mode similar relations are obtained for the corresponding variances, say $\tilde{X}$ and $\tilde{Y}$:

$$\Delta\tilde{X}^2(\theta) = \Delta\tilde{X}^2 \exp(-2\theta)\ \ , \qquad \Delta\tilde{Y}^2(\theta) = \Delta\tilde{Y}^2 \exp(2\theta)\ \ . \qquad (45)$$

For positive θ, squeezing then reduces the variances of the Y and $\tilde{X}$ variables, while the variances of the X and $\tilde{Y}$ variables grow by the same amount so to keep the uncertainty relations unchanged. This reflects, in terms of the A and B modes, the constancy of the difference $\mathcal{N}_{A_\kappa} - \mathcal{N}_{B_\kappa}$ against separate, but equal, changes of $\mathcal{N}_{A_\kappa}$ and $\mathcal{N}_{B_\kappa}$ (degeneracy of the states $|0(t)>$ *labelled* by different $\mathcal{N}_{A_\kappa}$, or different $\mathcal{N}_{B_\kappa}$, cf. Eq. (37).

In conclusion, the θ-set $\{\theta_\kappa(\mathcal{N}_\kappa)\}$, is nothing but the squeezing parameter classifying the squeezed coherent states in the hyperplane $(X, \tilde{X}; Y, \tilde{Y})$. Note that to different squeezed states (different θ-sets) are associated unitarily inequivalent representations of the ccr's in the infinite volume limit. Also note that in the limit $t \to \infty$ the variances of the variables Y and $\tilde{X}$ become infinity making them completely spread out.

Further details on the squeezing states and their relation with deformed algebraic structures in QFT can be found in refs. [28, 42, 43].

5 Quantum Brownian motion

By following Schwinger [20], the description of a Brownian particle of mass M moving in a potential U(x) with a damping resistance R, interacting with a thermal bath at temperature T is provided by [23, 24]

$$\mathcal{H}_{Brownian} = \mathcal{H} - \frac{ik_B TR}{\hbar}(x_+ - x_-)^2 \; . \quad (46)$$

Here $\mathcal{H}$ is given by Eq. (18) and the evolution equation for the density matrix is

$$i\hbar \frac{\partial(x_+|\rho(t)|x_-)}{\partial t} = \mathcal{H}(x_+ \, |\rho(t)| \, x_-) - (x_+ \, |N[\rho]| \, x_-) \; , \quad (47)$$

where $N[\rho] \approx (ik_B TR/\hbar)[x,[x,\rho]]$ describes the effects of the reservoir random thermal noise [23, 24].

In general the density operator in the above expression describes a mixed statistical state. The thermal bath contribution to the right hand side of Eq.(46), proportional to fluid temperature T, can be shown [24] to be equivalent to a white noise fluctuation source coupling the forward and backward motions according to

$$< y(t)y(t') >_{noise} = \frac{\hbar^2}{2Rk_B T}\delta(t - t'), \quad (48)$$

so that thermal fluctuations are always occurring in the difference $y = x_+ - x_-$ between forward in time and backward in time coordinates.

The correlation function for the random force f on the particle due to the bath is given by $G(t-s) = (i/\hbar) < f(t)f(s) >$. The retarded and advanced Greens functions are studied in ref. [23] and for brevity I omit here their discussion. The mechanical resistance is defined by $R = lim_{\omega \to 0} \mathcal{R}eZ(\omega + i0^+),$ with the mechanical impedance $Z(\zeta)$ (analytic in the upper half complex frequency plane $\mathcal{I}m \; \zeta > 0$) determined by the retarded Greens function $-i\zeta Z(\zeta) = \int_0^\infty dt G_{ret}(t) e^{i\zeta t}$. The time domain quantum noise in the fluctuating random force is $N(t-s) = (1/2) < f(t)f(s) + f(s)f(t) >$.

The interaction between the bath and the particle is evaluated by following Feynman and Vernon and one finds [23] for the real and the imaginary part of the action

$$\mathcal{R}e\mathcal{A}[x,y] = \int_{t_i}^{t_f} dt L, \quad (49)$$

$$\mathcal{I}m\mathcal{A}[x,y] = (1/2\hbar) \int_{t_i}^{t_f} \int_{t_i}^{t_f} dt ds N(t-s) y(t) y(s), \quad (50)$$

respectively, where L is defined in Eq. (20) for the given choice of $U(x_\pm)$ there adopted (without loss of generality).

I observe that at the classical level the "extra" coordinate y, is usually constrained to vanish. Note that $y(t) = 0$ is a true solution to Eqs. (22) so that the constraint is *not* in violation of the equations of motion. From Eqs. (49) and (50) one sees that *at quantum level nonzero y allows quantum noise effects arising from the imaginary part of the action*. On the contrary, in the classical "$\hbar \to 0$" limit nonzero y yields an "unlikely process" in view of the large imaginary part of the action implicit in Eq. (50). Thus, the meaning of the constraint $y = 0$ at the classical level is the one of avoiding such "unlikely process".

The rôle of the doubled y coordinate (the quantum β, or B mode in the discussion of the previous Section) is thus shown again to be absolutely crucial in the quantum regime. There it accounts for the quantum noise in the fluctuating random force in the system-environment coupling [23]: in the limit of $y \to 0$ (i.e. for $x_+ = x_-$) quantum effects are lost and the classical limit is obtained.

It is interesting to remark that the forward and backward in time velocity components $v_\pm = \dot{x}_\pm$ in the (x_+, x_-) plane

$$v_\pm = \frac{\partial \mathcal{H}}{\partial p_\pm} = \pm \frac{1}{M}(p_\pm \mp \frac{R}{2} x_\mp) \tag{51}$$

do not commute

$$[v_+, v_-] = i\hbar \frac{R}{M^2}, \tag{52}$$

and it is thus impossible to fix these velocities v_+ and v_- as being identical. Eq.(52) is similar to the usual commutation relations for the quantum velocities $\mathbf{v} = (\mathbf{p} - (e\mathbf{A}/c))/M$ of a charged particle moving in a magnetic field $\mathbf{B}$; i.e. $[v_1, v_2] = (i\hbar e B_3/M^2 c)$. Just as the magnetic field $\mathbf{B}$ induces an Aharonov-Bohm phase interference for the charged particle, the Brownian motion friction coefficient R induces an analogous phase interference between forward and backward motion which expresses itself as mechanical damping. Eq. (52) will be also discussed in connection with noncommutative geometry induced by quantum dissipation [40]. I will comment more on this in the next Section.

In the discussion above I have considered the low temperature limit: $T \ll T_\gamma$ where $k_B T_\gamma = \hbar\gamma = \frac{\hbar R}{2M}$. At high temperature, $T \gg T_\gamma$, the thermal bath motion suppresses the probability for $x_+ \neq x_-$ due to the thermal term $(k_B T R/\hbar)(x_+ - x_-)^2$ in Eq.(46) (cf. also Eq. (48)). By writing the diffusion coefficient $D = \frac{k_B T}{R}$ as

$$D = \frac{T}{T_\gamma}\left(\frac{\hbar}{2M}\right), \tag{53}$$

the condition for classical Brownian motion for high mass particles is that $D \gg (\hbar/2M)$, and the condition for quantum interference with low mass particles is that $D \ll (\hbar/2M)$. In colloidal systems, for example, classical Brownian motion for large particles would appear to dominate the motion. In a fluid at room temperature it is typically $D \sim (\hbar/2M)$ for a single atom, or, equivalently, $T \sim T_\gamma$, so that the rôle played by quantum mechanics, although perhaps not dominant, may be an important one in the Brownian motion.

6 Dissipative noncommutative plane

The harmonic oscillator on the noncommutative plane, the motion of a particle in an external magnetic field and the Landau problem on the noncommutative sphere are only few examples of systems whose noncommutative geometry has been studied in detail. Noncommutative geometries are also of interest in Cern-Simons gauge theories, the usual gauge theories and string theories, in gravity theory [44, 45]. Here I show that quantum dissipation induces noncommutative geometry in the (x_+, x_-) plane [40].

The velocity components $v_\pm = \dot{x}_\pm$ in the (x_+, x_-) plane are given Eq. (51). Similarly,

$$\dot{p}_\pm = -\frac{\partial \mathcal{H}}{\partial x_\pm} = \mp U'(x_\pm) \mp \frac{R v_\mp}{2} \ . \tag{54}$$

From Eqs.(51) and (54) it follows that

$$M\dot{v}_\pm + R v_\mp + U'(x_\pm) = 0 \ . \tag{55}$$

When the choice $U(x_\pm) = \frac{1}{2}\kappa x_\pm^2$ is made, these are equivalent to the equations Eq.(21) and (22). The classical equation of motion including dissipation thereby holds true if $x_+(t) \approx x_-(t) \approx x(t)$:

$$M\dot{v} + Rv + U'(x) = 0 \ . \tag{56}$$

If one defines

$$Mv_\pm = \hbar K_\pm, \tag{57}$$

then Eq.(52) gives

$$[K_+, K_-] = \frac{iR}{\hbar} \equiv \frac{i}{L^2} \ , \tag{58}$$

and a canonical set of conjugate position coordinates (ξ_+, ξ_-) may be defined by

$$\begin{aligned} \xi_\pm &= \mp L^2 K_\mp \\ [\xi_+, \xi_-] &= iL^2. \end{aligned} \tag{59}$$

Another independent canonical set of conjugate position coordinates (X_+, X_-) is defined by

$$\begin{aligned} x_+ = X_+ + \xi_+ \ \ &, \ \ x_- = X_- + \xi_- \\ [X_+, X_-] &= -iL^2. \end{aligned} \tag{60}$$

Note that $[X_a, \xi_b] = 0$, where $a = \pm$ and $b = \pm$.

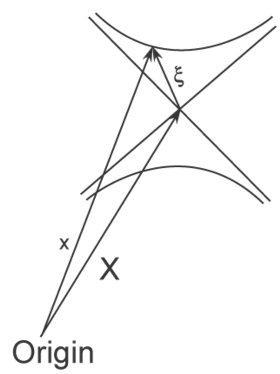

Fig. 2. The hyperbolic path of a particle moving in the $x = (x_+, x_-)$ plane. The noncommuting coordinate pairs $X = (X_+, X_-)$, which points from the origin to hyperbolic center, and $\xi = (\xi_+, \xi_-)$, which points from the center of the orbit to the position on the hyperbola, are shown. $x = X + \xi$.

The commutation relations Eqs.(59) and (60) characterize the noncommutative geometry in the plane (x_+, x_-). It is interesting to consider the case of pure friction in which the potential $U = 0$. Eqs.(18), (57) and (59) then imply

$$\mathcal{H}_{friction} = \frac{\hbar^2}{2M}(K_+^2 - K_-^2) = -\frac{\hbar^2}{2ML^4}(\xi_+^2 - \xi_-^2). \tag{61}$$

The equations of motion are

$$\dot{\xi}_\pm = \frac{i}{\hbar}\left[\mathcal{H}_{friction}, \xi_\pm\right] = -\frac{\hbar}{ML^2}\xi_\mp = -\frac{R}{M}\xi_\mp = -\Gamma\xi_\mp, \tag{62}$$

with the solution

$$\begin{pmatrix} \xi_+(t) \\ \xi_-(t) \end{pmatrix} = \begin{pmatrix} \cosh(\Gamma t) & -\sinh(\Gamma t) \\ -\sinh(\Gamma t) & \cosh(\Gamma t) \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}. \tag{63}$$

Eq.(63) describes the hyperbolic orbit

$$\xi_-(t)^2 - \xi_+(t)^2 = \frac{2L^2}{\hbar\Gamma}\mathcal{H}_{friction}\ . \tag{64}$$

The hyperbolae are defined by $(x-X)^2 - c^2(t-T)^2 = \Lambda^2$, where $\Lambda^2 = (\frac{mc}{\hbar}L^2)^2$, the hyperbolic center is at (X, cT) and one branch of the hyperbolae is a particle moving forward in time while the other branch is the same particle moving backward in time as an anti-particle.

Now I observe that a quantum phase interference of the Aharanov-Bohm type can always be associated with the noncommutative plane where

$$[X, Y] = iL^2\ , \tag{65}$$

with L denoting the geometric length scale in the plane. Suppose that a particle can move from an initial point in the plane to a final point in the plane via one of two paths, say $\mathcal{P}_1$ or $\mathcal{P}_2$. Since the paths start and finish at the same point, if one transverses the first path in a forward direction and the second path in a backward direction, then the resulting closed path encloses an area $\mathcal{A}$. The phase interference ϑ between these two points is determined by the difference between the actions for these two paths $\hbar\vartheta = \mathcal{S}(\mathcal{P}_1) - \mathcal{S}(\mathcal{P}_2)$, and I show below it may be written as

$$\vartheta = \frac{\mathcal{A}}{L^2}\ . \tag{66}$$

A physical realization of the mathematical noncommutative plane is present in every laboratory wherein a charged particle moves in a plane with a normal uniform magnetic field $\mathbf{B}$. For this case, there are two canonical pairs of position coordinates which do not commute: (i) the position $\mathbf{R}$ of the center of the cyclotron circular orbit and (ii) the radius vector ρ from the center of the circle to the charged particle position $\mathbf{r} = \mathbf{R} + \rho$ (Fig.3). The magnetic length scale of the noncommuting geometric coordinates is due to Landau [46],

$$L^2 = \frac{\hbar c}{eB} = \frac{\phi_0}{2\pi B} \quad \text{(magnetic)}. \tag{67}$$

Here ϕ_0 is the magnitude of the magnetic flux quantum associated with a charge e.

For motion at fixed energy one may (in classical mechanics) associate with each path $\mathcal{P}$ (in phase space) a phase space action integral

$$\mathcal{S}(\mathcal{P}) = \int_{\mathcal{P}} p_i dq^i. \tag{68}$$

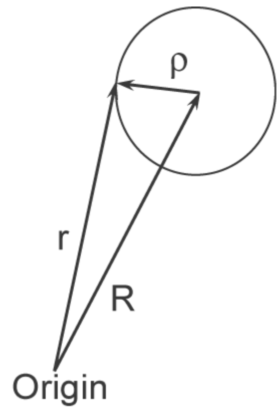

Fig. 3. A charge e moving in a circular cyclotron orbit. Noncommuting coordinate pairs are $\mathbf{R} = (X, Y)$, which points from the origin to the orbit center, and $\rho = (\rho_x, \rho_y)$, which points from the center of the orbit to the charge position $\mathbf{r} = \mathbf{R} + \rho$.

As said, the phase interference ϑ between the two paths $\mathcal{P}_1$ and $\mathcal{P}_2$ is determined by the action difference

$$\hbar\vartheta = \int_{\mathcal{P}_1} p_i dq^i - \int_{\mathcal{P}_2} p_i dq^i = \oint_{\mathcal{P}=\partial\Omega} p_i dq^i \tag{69}$$

wherein $\mathcal{P}$ is the closed path which goes from the initial point to the final point via path $\mathcal{P}_1$ and returns back to the initial point via $\mathcal{P}_2$. The closed $\mathcal{P}$ path may be regarded as the boundary of a two-dimensional surface Ω; i.e. $\mathcal{P} = \partial\Omega$. Stokes theorem yields

$$\vartheta = \frac{1}{\hbar}\oint_{\mathcal{P}=\partial\Omega} p_i dq^i = \frac{1}{\hbar}\int_{\Omega}(dp_i \wedge dq^i). \tag{70}$$

The quantum phase interference ϑ between two alternative paths is thereby proportional to an "area" $\mathcal{A}$ of a surface Ω in phase space $(p_1, \ldots, p_f; q^1, \ldots, q^f)$ as described by the right hand side of Eq.(70).

If one reverts to the operator formalism and writes the commutation Eq.(65) in the noncommutative plane as

$$[X, P_X] = i\hbar \quad \text{where} \quad P_X = \left(\frac{\hbar Y}{L^2}\right), \tag{71}$$

then back in the path integral formalism Eq.(70) reads

$$\vartheta = \frac{1}{\hbar}\int_{\Omega}(dP_X \wedge dX) = \frac{1}{L^2}\int_{\Omega}(dY \wedge dX) \tag{72}$$

and Eq.(66) is proved, i.e. the quantum phase interference between two alternative paths in the plane is determined by the noncommutative length scale L and the enclosed area $\mathcal{A}$.

I also remark that the existence of a phase interference is intimately connected to the zero point fluctuations in the coordinates; e.g. Eq.(65) implies a zero point uncertainty relation $\Delta X \Delta Y \geq (L^2/2)$.

Resorting back to Eq.(58) for the quantum dissipative case, i.e.

$$L^2 = \frac{\hbar}{R} \quad \text{(dissipative)}. \tag{73}$$

one then concludes that, provided $x_+ \neq x_-$, the quantum dissipative phase interference $\vartheta = \frac{\mathcal{A}}{L^2} = \frac{\mathcal{A}R}{\hbar}$ is associated with two paths in the noncommutative plane, starting at the same point $\mathcal{P}_1$ and ending to the same point $\mathcal{P}_2$ so to enclose the surface of area $\mathcal{A}$.

A comparison can be made between the noncommutative dissipative plane and the noncommutative Landau magnetic plane as shown in Fig.3. The circular orbit in Fig.3 for the magnetic problem is replaced by the hyperbolic orbit and it may be shown that the magnetic field is replaced by the electric field. The hyperbolic orbit in Fig.2 is reflected in the classical orbit for a charged particle moving along the x-axis in a uniform electric field. For more details on this comparison see [40].

Finally, I recall that the Lagrangian for the system of Eqs. (55) has been found [27] to be the same as the Lagrangian for three-dimensional topological massive Chern-Simons gauge theory in the infrared limit. It is also the same as for a Bloch electron in a solid which propagates along a lattice plane with a hyperbolic energy surface [27]. In the Chern-Simons case one has $\theta_{CS} = R/M = (\hbar/ML^2)$, with θ_{CS} the "topological mass parameter". In the Bloch electron case, $(eB/\hbar c) = (1/L^2)$, with B denoting the z-component of the applied external magnetic field. In ref. [27] it has been considered the symplectic structure for the system of Eqs. (55) in the case of strong damping $R \gg M$ (the so-called reduced case) in the Dirac constraint formalism as well as in the Faddeev and Jackiw formalism [47] and in both formalisms a non-zero Poisson bracket for the $(x_+,\ x_-)$ coordinates has been found.

Below I will consider the algebraic structure of the space of the physical states emergent from the doubling of the degrees of freedom discussed in the present and in the previous Section. Before that I will discuss thermal field theory in the following Section.

7 Thermal field theory

In this section I discuss the doubling of the degrees of freedom in connection with thermal field theory. Specifically, I will comment on the formalism of thermo field dynamics (TFD) [5, 8, 48]. In Section 8 it will be shown that the algebraic structure on which the TFD formalism is based is naturally provided by the q–deformed Hopf algebras for bosons and for fermions (usually called $h_q(1)$ and $h_q(1|1)$, respectively).

The central point in the TFD formalism is the possibility to express the statistical average $< \mathcal{A} >$ of an observable $\mathcal{A}$ as the expectation value in the temperature dependent vacuum $|0(\beta)>$:

$$< \mathcal{A} > \ \equiv \ \frac{Tr[\mathcal{A}\ e^{-\beta \mathcal{H}}]}{Tr[e^{-\beta \mathcal{H}}]} \ = \ < 0(\beta)|\mathcal{A}|0(\beta) > , \tag{74}$$

where $\mathcal{H} = H - \mu N$, with μ the chemical potential.

The first problem is therefore to construct a suitable temperature dependent state $|0(\beta)>$ which satisfies Eq. (74), namely

$$< 0(\beta)|\mathcal{A}|0(\beta) >= \frac{1}{Tr[e^{-\beta\mathcal{H}}]} \sum_n < n|\mathcal{A}|n > e^{-\beta E_n} \ , \tag{75}$$

for an arbitrary variable $\mathcal{A}$, with

$$\mathcal{H}|n >= E_n|n > \ , \quad < n|m >= \delta_{nm} \ . \tag{76}$$

Such a state cannot be constructed as long as one remains in the original Fock space $\{|n >\}$. To see this, let me closely follow [8]. One can expand $|0(\beta)>$ in terms of $|n>$ as

$$|0(\beta) >= \sum_n f_n(\beta)|n > \ . \tag{77}$$

Then, use of this equation into (75) gives

$$f_n^*(\beta) f_m(\beta) = \frac{1}{Tr[e^{-\beta\mathcal{H}}]} e^{-\beta E_n} \delta_{nm} \ , \tag{78}$$

which is impossible to be satisfied by c-number functions $f_n(\beta)$. However, Eq. (78) can be regarded as the orthogonality condition in a Hilbert space in which the expansion coefficient $f_n(\beta)$ is a vector. In order to realize such a representation it is convenient to introduce a dynamical system identical to the one under study, namely to double the given system. The quantities associated with the doubled system are denoted by the tilde in the usual notation of TFD [8]. Thus the tilde-system is characterized by the Hamiltonian $\tilde{H}$ and the states are denoted by $|\tilde{n}>$, with

$$\tilde{\mathcal{H}}|\tilde{n} >= E_n|\tilde{n} > \ , \quad < \tilde{n}|\tilde{m} >= \delta_{nm} \ . \tag{79}$$

where E_n is the same as the one appearing in Eq. (76) by definition. It is also assumed that non-tilde and tilde operators are commuting (anti-commuting) boson (fermion) operators. One then considers the space spanned by the direct product $|n > \otimes |\tilde{m} > \equiv |n, \tilde{m}>$. The matrix element of a bose-like operator $\mathcal{A}$ is then

$$< \tilde{m}, n|\mathcal{A}|n', \tilde{m}' >=< n|\mathcal{A}|n' > \delta_{mm'} \ , \tag{80}$$

and the one of the corresponding $\tilde{\mathcal{A}}$ is

$$< \tilde{m}, n|\tilde{\mathcal{A}}|n', \tilde{m}' >=< \tilde{m}|\tilde{\mathcal{A}}|\tilde{m}' > \delta_{nn'} \ . \tag{81}$$

In TFD it turns out to be convenient to identify

$$< m|\mathcal{A}|n >=< \tilde{n}|\tilde{\mathcal{A}}^\dagger|\tilde{m} > \ . \tag{82}$$

Eq. (78) is satisfied if one defines

$$f_n(\beta) = \frac{1}{\sqrt{Tr[e^{-\beta\mathcal{H}}]}} e^{\frac{-\beta E_n}{2}} |\tilde{n} > \ , \tag{83}$$

and Eq. (75) is obtained by using the definition (83) in $|0(\beta)>$ given by (77):

$$|0(\beta)>=\frac{1}{\sqrt{Tr[e^{-\beta\mathcal{H}}]}}\sum_n e^{\frac{-\beta E_n}{2}}|n,\tilde{n}> \ . \tag{84}$$

The vectors $|n>$ and $|\tilde{n}>$ thus appear as a pair in $|0(\beta)>$. I remark that the formal rôle of the "doubled" states $|\tilde{n}>$ is merely to pick up the diagonal matrix elements of $\mathcal{A}$. In this connection, thinking of the rôle of the environment, which is able to reduce the system density matrix to its diagonal form in the QM decoherence processes [12], it is remarkable that the doubled degrees of freedom in TFD are indeed susceptible of being interpreted as the environment degrees of freedom, as better specified in the following.

It is useful to consider, as an example, the case of the number operator. Let $\mathcal{A} \equiv N = a^\dagger a$. For definiteness I consider the boson case. Then the statistical average of N is the Bose-Einstein distribution $f_B(\omega)$, where ω denotes the mode energy, $H = \omega a^\dagger a$,

$$< N > \ \equiv \ \frac{Tr[N\ e^{-\beta H}]}{Tr[e^{-\beta H}]} \ = \ < 0(\beta)|N|0(\beta)>=\frac{1}{e^{\beta\omega}-1}=f_B(\omega)\ . \tag{85}$$

One then can show [8] that, by setting

$$u(\beta)\equiv\ \sqrt{1+f_B(\omega)},\quad v(\beta)\equiv\ \sqrt{f_B(\omega)}\ , \tag{86}$$

$$u^2(\beta)-v^2(\beta)=1\ , \tag{87}$$

so that

$$u(\beta)=\cosh\theta(\beta)\ ,\quad v(\beta)=\sinh\theta(\beta)\ , \tag{88}$$

and defining

$$\mathcal{G}\equiv -i(a^\dagger\tilde{a}^\dagger - a\tilde{a})\ , \tag{89}$$

the state $|0(\beta)>$ is formally given (at finite volume) by

$$|0(\beta)>=e^{i\theta(\beta)\mathcal{G}}|0>=\frac{1}{u(\beta)}\exp\left(\frac{v(\beta)}{u(\beta)}\right)a^\dagger\tilde{a}^\dagger|0> \ . \tag{90}$$

It is clear that the state $|0(\beta)>$ is not annihilated by a and $\tilde{a}$. However, it is annihilated by the "new" set of operators $a(\theta)$ and $\tilde{a}(\theta)$,

$$a(\theta)|0(\beta)>=0=\tilde{a}(\theta)|0(\beta)> \ , \tag{91}$$

with

$$\begin{aligned} a(\theta) &= \exp(i\theta\mathcal{G})\ a\ \exp(-i\theta\mathcal{G}) = a\cosh\theta-\tilde{a}^\dagger\sinh\theta\ ,\\ \tilde{a}(\theta) &= \exp(i\theta\mathcal{G})\ \tilde{a}\ \exp(-i\theta\mathcal{G})\ = \tilde{a}\cosh\theta-a^\dagger\sinh\theta\ , \end{aligned} \tag{92}$$

$$[a(\theta),a^\dagger(\theta)]=1\ ,\quad [\tilde{a}(\theta),\tilde{a}^\dagger(\theta)]=1\ . \tag{93}$$

All other commutators are equal to zero and $a(\theta)$ and $\tilde{a}(\theta)$ commute among themselves. Eqs. (92) are nothing but the Bogoliubov transformations of the $(a,\tilde{a})$ pair into a new set of creation, annihilation operators. I will show in Section 8 that the Bogoliubov-transformed operators $a(\theta)$ and $\tilde{a}(\theta)$ are linear combinations of the deformed coproduct operators.

The state $|0(\beta)>$ is not the vacuum (zero energy eigenstate) of H and of $\tilde{H}$. It is, however, the zero energy eigenstate for the "Hamiltonian" $\hat{H}$, $\hat{H}|0(\beta)>=0$, with

$$\hat{H} \equiv H - \tilde{H} = \omega(a^\dagger a - \tilde{a}^\dagger \tilde{a}). \ , \qquad (94)$$

The state $|0(\beta)>$ is called the thermal vacuum.

I note that in the boson case $J_1 \equiv \frac{1}{2}(a^\dagger \tilde{a}^\dagger + a\tilde{a})$ together with $J_2 \equiv \frac{1}{2}\mathcal{G}$ and $J_3 \equiv \frac{1}{2}(N + \tilde{N} + 1)$ close the algebra $su(1,1)$. Moreover, $\frac{\delta}{\delta\theta}(N(\theta) - \tilde{N}(\theta)) = 0$, with $(N(\theta) - \tilde{N}(\theta)) \equiv (a^\dagger(\theta)a(\theta) - \tilde{a}^\dagger(\theta)\tilde{a}(\theta))$, consistently with the fact that $\frac{1}{4}(N - \tilde{N})^2$ is the $su(1,1)$ Casimir operator.

In the fermion case $J_1 \equiv \frac{1}{2}\mathcal{G}$, $J_2 \equiv \frac{1}{2}(a^\dagger \tilde{a}^\dagger + a\tilde{a})$ and $J_3 \equiv \frac{1}{2}(N + \tilde{N} - 1)$ close the algebra $su(2)$. Also in this case $\frac{\delta}{\delta\theta}(N(\theta) - \tilde{N}(\theta)) = 0$, with $(N(\theta) - \tilde{N}(\theta)) \equiv (a^\dagger(\theta)a(\theta) - \tilde{a}^\dagger(\theta)\tilde{a}(\theta))$, again consistently with the fact that $\frac{1}{4}(N - \tilde{N})^2$ is related to the $su(2)$ Casimir operator.

Summarizing, the vacuum state for $a(\theta)$ and $\tilde{a}(\theta)$ is formally given (at finite volume) by

$$|0(\theta)> \ = \ \exp\ (i\theta\mathcal{G})\ \ |0,0> \ = \ \sum_n c_n(\theta)\,|n,\tilde{n}> \ , \qquad (95)$$

with $n, \tilde{n} = 0, .., \infty$ for bosons and $n, \tilde{n} = 0, 1$ for fermions, and it appears therefore to be an $SU(1,1)$ or $SU(2)$ generalized coherent state [33], respectively for bosons or for fermions.

In the infinite volume limit $|0(\theta)>$ becomes orthogonal to $|0,0>$ and we have that the whole Hilbert space $\{|0(\theta)>\}$, constructed by operating on $|0(\theta)>$ with $a^\dagger(\theta)$ and $\tilde{a}^\dagger(\theta)$, is asymptotically (i.e. in the infinite volume limit) orthogonal to the space generated over $\{|0,0>\}$. In general, for each value of $\theta(\beta)$, i.e. for each value of the temperature $T = \frac{1}{k_B\beta}$, one obtains in the infinite volume limit a representation of the canonical commutation relations unitarily inequivalent to the others, associated with different values of T. In other words, the parameter $\theta(\beta)$ (or the temperature T) acts as a label for the inequivalent representations [25].

The TFD formalism is a fully developed QFT formalism [5, 8, 48] and it has been applied to a rich set of problems of physical interest, in condensed matter physics, high energy physics, quantum optics, etc. (see [5, 8, 17, 18, 25, 28, 37, 48] and references therein quoted). I will show in Section 8 that the doubling of the degrees of freedom on which the TFD formalism is based finds its natural realization in the coproduct map.

Let me recall the so-called ″tilde-conjugation rules″ which are defined in TFD. For any two bosonic (respectively, fermionic) operators $\mathcal{O}$ and $\mathcal{O}'$ and any two c-numbers α and β the tilde-conjugation rules of TFD are postulated to be the following [8]:

$$(\mathcal{O}\mathcal{O}')\tilde{} = \tilde{\mathcal{O}}\tilde{\mathcal{O}}' \ , \qquad (96)$$

$$(\alpha\mathcal{O} + \beta\mathcal{O}')\tilde{} = \alpha^*\tilde{\mathcal{O}} + \beta^*\tilde{\mathcal{O}}' \ , \qquad (97)$$

$$(\mathcal{O}^\dagger)\tilde{} = \tilde{\mathcal{O}}^\dagger \ , \qquad (98)$$

$$(\tilde{\mathcal{O}})\tilde{} = \mathcal{O} \ . \qquad (99)$$

According to (96) the tilde-conjugation does not change the order among operators. Furthermore, it is required that tilde and non-tilde operators are mutually commuting (or anti-commuting) operators and that the thermal vacuum $|0(\beta)>$ is invariant under tilde-conjugation:

$$[\mathcal{O}, \tilde{\mathcal{O}}']_\mp = 0 = [\mathcal{O}, \tilde{\mathcal{O}}'^\dagger]_\mp \ , \qquad (100)$$

$$|0(\beta)>\tilde{} = |0(\beta)> \quad . \tag{101}$$

In order to use a compact notation it is useful to introduce the label σ defined by $\sqrt{\sigma} \equiv +1$ for bosons and $\sqrt{\sigma} \equiv +i$ for fermions. I shall therefore simply write commutators as $[\mathcal{O}, \mathcal{O}']_{-\sigma} \doteq \mathcal{O}\mathcal{O}' - \sigma\mathcal{O}'\mathcal{O}$, and $(1 \otimes \mathcal{O})(\mathcal{O}' \otimes 1) \equiv \sigma(\mathcal{O}' \otimes 1)(1 \otimes \mathcal{O})$, without further specification of whether $\mathcal{O}$ and $\mathcal{O}'$ (which are equal to a, $a^\dagger$ in all possible ways) are fermions or bosons.

Upon identifying from now on $a_1 \equiv a$, $a_1^\dagger \equiv a^\dagger$, one easily checks that the TFD tilde-operators (consistent with (96) – (101)) are straightforwardly recovered by setting $a_2 \equiv \tilde{a}$, $a_2^\dagger \equiv \tilde{a}^\dagger$. In other words, according to such identification, it is the action of the $1 \leftrightarrow 2$ permutation π: $\pi a_i = a_j$, $i \neq j$, $i, j = 1, 2$, that realizes the operation of ″tilde-conjugation″ defined in (96 - 99):

$$\pi a_1 = \pi(a \otimes \mathbf{1}) = \mathbf{1} \otimes a = a_2 \equiv \tilde{a} \equiv (a)\tilde{} \tag{102}$$

$$\pi a_2 = \pi(\mathbf{1} \otimes a) = a \otimes \mathbf{1} = a_1 \equiv a \equiv (\tilde{a})\tilde{} \ . \tag{103}$$

In particular, since the permutation π is involutive, also tilde-conjugation turns out to be involutive, as in fact required by the rule (99). Notice that, as $(\pi a_i)^\dagger = \pi(a_i{}^\dagger)$, it is also $((a_i)\tilde{}\ )^\dagger = ((a_i)^\dagger)\tilde{}$, i.e. tilde-conjugation commutes with hermitian conjugation. Furthermore, from (102)-(103), one has

$$(ab)\tilde{} = [(a \otimes \mathbf{1})(b \otimes \mathbf{1})]\tilde{} = (ab \otimes \mathbf{1})\tilde{} = \mathbf{1} \otimes ab = (\mathbf{1} \otimes a)(\mathbf{1} \otimes b) = \tilde{a}\tilde{b} \ . \tag{104}$$

Rules (98) and (96) are thus obtained. (100) is insured by the σ-commutativity of a_1 and a_2. The vacuum of TFD, $|0(\beta)>$, is a condensed state of equal number of tilde and non-tilde particles [8], thus (101) requires no further conditions: Eqs. (102)-(103) are sufficient to show that the rule (101) is satisfied.

TFD appears equipped with a set of canonically conjugate ″thermal″ variables: θ and $p_\theta \equiv -i\dfrac{\delta}{\delta\theta}$. p_θ can be regarded as the momentum operator ″conjugate″ to the ″thermal degree of freedom″ θ. The notion of thermal degree of freedom [48] thus acquires formal definiteness in the sense of the canonical formalism. It is remarkable that the "conjugate thermal momentum" p_θ generates transitions among inequivalent (in the infinite volume limit) representations: $\exp(i\bar{\theta}p_\theta)\ |0(\theta)> = |0(\theta + \bar{\theta})>$. Notice that derivative with respect to the θ parameter is actually a derivative with respect to the system temperature T. This sheds some light on the rôle of θ in thermal field theories for non-equilibrium systems and phase transitions. I shall comment more on this point in the following Section.

Finally, when the proper field description is taken into account, a and $\tilde{a}$ carry dependence on the momentum $\mathbf{k}$. The Bogoliubov transformation analogously, should be thought of as inner automorphism of the algebra $su(1,1)_\mathbf{k}$ (or $su(2)_\mathbf{k}$). This shows that one is globally dealing with $\bigoplus_\mathbf{k} su(1,1)_\mathbf{k}$ (or $\bigoplus_\mathbf{k} su(2)_\mathbf{k}$). Therefore one is lead to consider $\mathbf{k}$-dependence also for the θ parameter.

As a final comment, I observe that the "analogies" with the formalism of quantum dissipation presented in Section 3.1 are evident.

8 The q–deformed Hopf algebra and QFT

In this Section I want to point out that the doubling of the degrees of freedom is intimately related to the structure of the space of the states in QFT [9]. This brings us to consider the q–deformed Hopf algebra [6, 7].

One key ingredient of Hopf algebra [7] is the coproduct operation, i.e. the operator doubling implied by the coalgebra. The coproduct operation is indeed a map $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ which duplicates the algebra $\mathcal{A}$. Coproducts are commonly used in the familiar addition of energy, momentum, angular momentum and of other so-called primitive operators. The coproduct of a generic operator $\mathcal{O}$ is a homomorphism defined as $\Delta\mathcal{O} = \mathcal{O} \otimes \mathbf{1} + \mathbf{1} \otimes \mathcal{O} \equiv \mathcal{O}_1 + \mathcal{O}_2$, with $\mathcal{O} \in \mathcal{A}$. Since additivity of observables such as energy, momentum, angular momentum, etc. is an essential requirement, the coproduct, and therefore the Lie-Hopf algebra structure, appears to provide an essential algebraic tool in QM and in QFT.

The systems discussed in the Sections above, where the duplication of the degrees of freedom has revealed to be central, are thus natural candidates to be described by the Lie-Hopf algebra. The remarkable result holds [9] according to which the infinitely many ui representations of the ccr, whose existence characterizes QFT, are classified by use of the *q–deformed* Hopf algebra. Quantum deformations of Hopf algebra have thus a deeply non-trivial physical meaning in QFT.

In the following I consider boson operators. The discussion and the conclusions can be easily extended to the case of fermion operators [9]. For notational simplicity I will omit the momentum suffix κ.

The bosonic algebra $h(1)$ is generated by the set of operators $\{a, a^\dagger, H, N\}$ with commutation relations:

$$[\, a \,,\, a^\dagger \,] = \; 2H \,, \quad [\; N \,,\, a \,] = -a \,, \quad [\; N \,,\, a^\dagger \,] = a^\dagger \,, \quad [\; H \,,\, \bullet \,] = 0 \,. \tag{105}$$

H is a central operator, constant in each representation. The Casimir operator is given by $\mathcal{C} = 2NH - a^\dagger a$. $h(1)$ is an Hopf algebra and is therefore equipped with the coproduct operation, defined by

$$\Delta a = a \otimes \mathbf{1} + \mathbf{1} \otimes a \equiv a_1 + a_2 \;, \qquad \Delta a^\dagger = a^\dagger \otimes \mathbf{1} + \mathbf{1} \otimes a^\dagger \equiv a_1^\dagger + a_2^\dagger \;, \tag{106}$$

$$\Delta H = H \otimes \mathbf{1} + \mathbf{1} \otimes H \equiv H_1 + H_2 \;, \quad \Delta N = N \otimes \mathbf{1} + \mathbf{1} \otimes N \equiv N_1 + N_2 \;. \tag{107}$$

Note that $[a_i, a_j] = [a_i, a_j^\dagger] = 0,\; i, j = 1, 2,\; i \neq j$. The coproduct provides the prescription for operating on two modes. As mentioned, one familiar example of coproduct is the addition of the angular momentum J^α, $\alpha = 1, 2, 3$, of two particles: $\Delta J^\alpha = J^\alpha \otimes \mathbf{1} + \mathbf{1} \otimes J^\alpha \equiv J_1^\alpha + J_2^\alpha,\; J^\alpha \in su(2)$.

The q-deformation of $h(1)$ is the Hopf algebra $h_q(1)$:

$$[\, a_q \,,\, a_q^\dagger \,] = \; [2H]_q \,, \quad [\; N \;,\, a_q \,] = -a_q \,, \quad [\; N \,,\, a_q^\dagger \,] = a_q^\dagger, \quad [\; H \;,\, \bullet \,] = 0 \,, \tag{108}$$

where $N_q \equiv N$ and $H_q \equiv H$. The Casimir operator $\mathcal{C}_q$ is given by $\mathcal{C}_q = N[2H]_q - a_q^\dagger a_q$, where $[x]_q = \dfrac{q^x - q^{-x}}{q - q^{-1}}$. The deformed coproduct is defined by

$$\Delta a_q = a_q \otimes q^H + q^{-H} \otimes a_q \;, \qquad \Delta a_q^\dagger = a_q^\dagger \otimes q^H + q^{-H} \otimes a_q^\dagger \;, \tag{109}$$

$$\Delta H = H \otimes \mathbf{1} + \mathbf{1} \otimes H \;, \quad \Delta N = N \otimes \mathbf{1} + \mathbf{1} \otimes N \;, \tag{110}$$

whose algebra is isomorphic with (108): $[\Delta a_q, \Delta a_q^\dagger] = [2\Delta H]_q$, etc. . Note that $h_q(1)$ is a structure different from the commonly considered q-deformation of the harmonic oscillator [49] that does not have a coproduct and thus cannot allow for the duplication of the state space.

I denote by $\mathcal{F}_1$ the single mode Fock space, i.e. the fundamental representation $H = 1/2$, $\mathcal{C} = 0$. In such a representation $h(1)$ and $h_q(1)$ coincide as it happens for $su(2)$ and $su_q(2)$ for the spin-$\frac{1}{2}$ representation. The differences appear in the coproduct and in the higher spin representations.

As customary, I require that a and $a^\dagger$, and a_q and ${a_q}^\dagger$, are adjoint operators. This implies that q can only be real (or of modulus one in the fermionic case. In the two mode Fock space $\mathcal{F}_2 = \mathcal{F}_1 \otimes \mathcal{F}_1$, for $|q| = 1$, the hermitian conjugation of the coproduct must be supplemented by the inversion of the two spaces for consistency with the coproduct isomorphism).

Summarizing, one can write for both bosons (and fermions) on $\mathcal{F}_2 = \mathcal{F}_1 \otimes \mathcal{F}_1$:

$$\Delta a = a_1 + a_2 \ , \quad \Delta a^\dagger = a_1^\dagger + a_2^\dagger \ , \tag{111}$$

$$\Delta a_q = a_1 q^{1/2} + q^{-1/2} a_2 \ , \quad \Delta a_q^\dagger = a_1^\dagger q^{1/2} + q^{-1/2} a_2^\dagger \ , \tag{112}$$

$$\Delta H = 1, \quad \Delta N = N_1 + N_2 \ . \tag{113}$$

Now, the key point is [9] that the full set of infinitely many unitarily inequivalent representations of the ccr in QFT are classified by use of the q–deformed Hopf algebra. Since, as well known, the Bogolubov transformations relate different (i.e. unitary inequivalent) representations, it is sufficient to show that the Bogolubov transformations are directly obtained by use of the deformed copodruct operation. I consider therefore the following operators (cf. (109) with $q(\theta) \equiv e^{2\theta}$ and $H = 1/2$):

$$\alpha_{q(\theta)} \equiv \frac{\Delta a_q}{\sqrt{[2]_q}} = \frac{1}{\sqrt{[2]_q}} (e^\theta a_1 + e^{-\theta} a_2) \ , \tag{114}$$

$$\beta_{q(\theta)} \equiv \frac{1}{\sqrt{[2]_q}} \frac{\delta}{\delta\theta} \Delta a_q = \frac{2q}{\sqrt{[2]_q}} \frac{\delta}{\delta q} \Delta a_q = \frac{1}{\sqrt{[2]_q}} (e^\theta a_1 - e^{-\theta} a_2) \ , \tag{115}$$

and h.c.. A set of commuting operators with canonical commutation relations is given by

$$\alpha(\theta) \equiv \frac{\sqrt{[2]_q}}{2\sqrt{2}} [\alpha_{q(\theta)} + \alpha_{q(-\theta)} - \beta^\dagger_{q(\theta)} + \beta^\dagger_{q(-\theta)}] \ , \tag{116}$$

$$\beta(\theta) \equiv \frac{\sqrt{[2]_q}}{2\sqrt{2}} [\beta_{q(\theta)} + \beta_{q(-\theta)} - \alpha^\dagger_{q(\theta)} + \alpha^\dagger_{q(-\theta)}] \ . \tag{117}$$

and h.c. One then introduces [9]

$$A(\theta) \equiv \frac{1}{\sqrt{2}} (\alpha(\theta) + \beta(\theta)) = A \cosh\theta - B^\dagger \sinh\theta \ , \tag{118}$$

$$B(\theta) \equiv \frac{1}{\sqrt{2}} (\alpha(\theta) - \beta(\theta)) = B \cosh\theta - A^\dagger \sinh\theta \ , \tag{119}$$

with

$$[A(\theta), A^\dagger(\theta)] = 1 \ , \quad [B(\theta), B^\dagger(\theta)] = 1 \ . \tag{120}$$

All other commutators are equal to zero and $A(\theta)$ and $B(\theta)$ commute among themselves. Eqs. (118) and (119) are nothing but the Bogolubov transformations for the (A, B) pair (see the corresponding transformations, e.g. in the case of the dho, Eqs. (32) and (33)). In other words, Eqs. (118), (119) show that the Bogolubov-transformed operators $A(\theta)$ and $B(\theta)$ are linear combinations of the coproduct operators defined in terms of the deformation parameter $q(\theta)$ and of their θ-derivatives.

From this point on one can re-obtain the results discussed in the previous Sections, for example for the dho provided one sets $\theta \equiv \Gamma t$.

The generator of (118) and (119) is $\mathcal{G} \equiv -i(A^\dagger B^\dagger - AB)$:

$$- i\frac{\delta}{\delta\theta}A(\theta) = [\mathcal{G}, A(\theta)] \ , \quad - i\frac{\delta}{\delta\theta}B(\theta) = [\mathcal{G}, B(\theta)] \ , \tag{121}$$

and h.c.. Compare this generator with H_I in Eq. (28).

Let $|0\rangle \equiv |0\rangle \otimes |0\rangle$ denote the vacuum annihilated by A and B, $A|0\rangle = 0 = B|0\rangle$. By introducing the suffix κ (till now omitted for simplicity), at finite volume V one obtains

$$|0(\theta)\rangle \ = e^{i\sum_\kappa \theta_\kappa \mathcal{G}_\kappa}|0\rangle \ = \prod_k \frac{1}{\cosh\theta_k} e^{\tanh\theta_k A_k^\dagger B_k^\dagger}|0\rangle \ , \tag{122}$$

to be compared with Eq. (35). θ denotes the set $\{\theta_\kappa = \frac{1}{2}\ln q_\kappa, \forall\kappa\}$ and $\langle 0(\theta)|0(\theta)\rangle = 1$. The underlying group structure is $\bigotimes_\kappa SU(1,1)_\kappa$ and the vacuum $|0(\theta)\rangle$ is an $SU(1,1)$ generalized coherent state [33]. The q–deformed Hopf algebra is thus intrinsically related to coherence and to the vacuum structure in QFT.

In the infinite volume limit, the number of degrees of freedom becomes uncountable infinite, and thus one obtains [5, 8, 25] $\langle 0(\theta)|0(\theta')\rangle \to 0 \quad \text{as } V \to \infty, \quad \forall\ \theta, \theta',\ \theta \neq \theta'$. By denoting with $\mathcal{H}_\theta$ the Hilbert space with vacuum $|0(\theta)\rangle$, $\mathcal{H}_\theta \equiv \{|0(\theta)\rangle\}$, this means that $\mathcal{H}_\theta$ and $\mathcal{H}_{\theta'}$ become unitarily inequivalent. In this limit, the "points" of the space $\mathcal{H} \equiv \{\mathcal{H}_\theta,\ \forall\ \theta\}$ of the infinitely many ui representations of the ccr are labelled by the deformation parameter θ [9, 25]. The space $\mathcal{H} \equiv \{\mathcal{H}_\theta,\ \forall\ \theta\}$ is called the space of the representations.

I note that $p_\theta \equiv -i\frac{\delta}{\delta\theta}$ can be regarded [9] as the momentum operator "conjugate" to the "degree of freedom" θ. For an assigned fixed value $\bar\theta$, it is

$$e^{i\bar\theta p_\theta}A(\theta) = e^{i\bar\theta\mathcal{G}}A(\theta)e^{-i\bar\theta\mathcal{G}} = A(\theta + \bar\theta), \tag{123}$$

and similarly for $B(\theta)$.

It is interesting to consider the case of time–dependent deformation parameter. This immediately relates to the dissipative systems considered in the previous Sections. The Heisenberg equation for $A(t, \theta(t))$ is

$$-i\dot A(t,\theta(t)) = -i\frac{\delta}{\delta t}A(t,\theta(t)) - i\frac{\delta\theta}{\delta t}\frac{\delta}{\delta\theta}A(t,\theta(t)) =$$

$$[H, A(t,\theta(t))] + \frac{\delta\theta}{\delta t}[\mathcal{G}, A(t,\theta(t))] = [H + Q, A(t,\theta(t))] \, , \tag{124}$$

and $Q \equiv \frac{\delta\theta}{\delta t}\mathcal{G}$ plays the role of the heat–term in dissipative systems. H is the Hamiltonian responsible for the time variation in the explicit time dependence of $A(t, \theta(t))$. $H + Q$ can be therefore identified with the free energy [25]: variations in time of the deformation parameter involve dissipation. In thermal theories and in dissipative systems the doubled modes B play the role of the thermal bath or environment.

Summarizing, QFT is characterized by the existence of ui representations of the ccr [3] which are related among themselves by the Bogoliubov transformations.

These, as seen above, are obtained as linear combinations of the deformed coproduct maps which express the doubling of the degrees of freedom. Therefore one may conclude that the intrinsic algebraic structure of QFT (independent of the specificity of the system under study) is the one of the q–deformed Hopf algebra. The ui representations existing in QFT are related and labelled by means of such an algebraic structure.

It should be stressed that the coproduct map is also essential in QM in order to deal with a many mode system (typically, with identical particles). However, in QM all the representations of the ccr are unitarily equivalent and therefore the Bogoliubov transformations induce unitary transformations among the representations, thus preserving their physical content. The q–deformed Hopf algebra therefore does not have that physical relevance in QM, which it has, on the contrary, in QFT. Here, the representations of the ccr, related through Bogoliubov representations, are unitarily *inequivalent* and therefore physically inequivalent: they represent different physical phases of the system corresponding to different boundary conditions, such as, for example, the system temperature. Typical examples are the superconducting and the normal phase, the ferromagnetic and the non-magnetic (i.e. zero magnetization) phase, the crystal and the gaseous phase, etc.. The physical meaning of the deformation parameter q in terms of which ui representations are labelled is thus recognized.

When the above discussion is applied to non-equilibrium (e.g. thermal and/or dissipative) field theories it appears that the couple of conjugate variables θ and $p_\theta \equiv -i\frac{\partial}{\partial\theta}$, with $\theta = \theta(\beta(t))$ $(\beta(t) = \frac{1}{k_B T(t)})$, related to the q–deformation parameter, describe trajectories in the space $\mathcal{H}$ of the representations. In [10] it has been shown that there is a symplectic structure associated to the "degrees of freedom" θ and that the trajectories in the $\mathcal{H}$ space may exhibit properties typical of chaotic trajectories in classical nonlinear dynamics. I will discuss this in the following. In the next Section I present further characterizations of the vacuum structure of the ui representations in QFT.

9 Entropy as a measure of the entanglement

In Section 3 I have shown that the time evolution of the state $|0(t)\rangle$ is actually controlled by the entropy variations (cf. Eq. (41)). I will shortly comment on the entropy in this Section from a more general point of view, also in connection with entanglement of the $A - B$ modes, since it appears as a structural aspect of QFT related with the existence of the ui representations of the ccr.

The state $|0(\theta)\rangle$ may be written as:

$$|0(\theta)\rangle = \exp\left(-\frac{1}{2}S_A\right)|\mathcal{I}\rangle = \exp\left(-\frac{1}{2}S_B\right)|\mathcal{I}\rangle\ , \tag{125}$$

$$S_A \equiv -\sum_\kappa \left\{A_\kappa^\dagger A_\kappa \ln\sinh^2\theta_\kappa - A_\kappa A_\kappa^\dagger \ln\cosh^2\theta_\kappa\right\}\ . \tag{126}$$

Here $|\mathcal{I}\rangle \equiv \exp\left(\sum_\kappa A_\kappa^\dagger B_\kappa^\dagger\right)|0\rangle$ and S_B is given by an expression similar to S_A, with B_κ and $B_\kappa^\dagger$ replacing A_κ and $A_\kappa^\dagger$, respectively. I simply write S for either S_A or S_B. I can also write [5, 8, 25]:

$$|0(\theta)\rangle = \sum_{n=0}^{+\infty} \sqrt{W_n}\,(|n\rangle \otimes |n\rangle) \ , \tag{127}$$

$$W_n = \prod_k \frac{\sinh^{2n_k}\theta_k}{\cosh^{2(n_k+1)}\theta_k} \ , \tag{128}$$

with n denoting the set $\{n_\kappa\}$ and with $0 < W_n < 1$ and $\sum_{n=0}^{+\infty} W_n = 1$. Then

$$\langle 0(\theta)|S_A|0(\theta)\rangle = \sum_{n=0}^{+\infty} W_n ln W_n \ , \tag{129}$$

which confirms that S can be interpreted as the entropy operator [5, 8, 25].

The state $|0(\theta)\rangle$ in Eq. (122) can be also written as

$$\begin{aligned} |0(\theta)\rangle \ &= \left(\prod_k \frac{1}{\cosh \theta_k} \right) \\ &\times \left(|0\rangle \otimes |0\rangle + \sum_k \tanh \theta_k \,(|A_k\rangle \otimes |B_k\rangle) + \ldots \right) , \end{aligned} \tag{130}$$

which clearly cannot be factorized into the product of two single-mode states. There is thus entanglement between the modes A and B: $|0(\theta)\rangle$ is an entangled state. Eq. (127) and (129) then show that S provides a measure of the degree of entanglement.

I remark that the entanglement is truly realized in the infinite volume limit where

$$\langle 0(\theta)|0\rangle = e^{-\frac{V}{(2\pi)^3}\int d^3\kappa \,\ln\cosh\theta_\kappa} \underset{V\to\infty}{\longrightarrow} 0 \ , \tag{131}$$

provided $\int d^3\kappa \ln\cosh\theta_\kappa$ is not identically zero. The probability of having the component state $|n\rangle \otimes |n\rangle$ in the state $|0(\theta)\rangle$ is W_n. Since W_n is a decreasing monotonic function of n, the contribution of the states $|n\rangle \otimes |n\rangle$ would be suppressed for large n at finite volume. In that case, the transformation induced by the unitary operator $G^{-1}(\theta) \equiv \exp(-i\sum_\kappa \theta_\kappa \mathcal{G}_\kappa)$ could disentangle the A and B sectors. However, this is not the case in the infinite volume limit, where the summation extends to an infinite number of components and Eq. (131) holds (in such a limit Eq. (122) is only a formal relation since $G^{-1}(\theta)$ does not exist as a unitary operator)[19].

It is interesting to note that, although the mode B is related with quantum noise effects (cf. the discussion in Section 5), nevertheless the $A - B$ entanglement is not affected by such noise effects. The robustness of the entanglement is rooted in the fact that, once the infinite volume limit is reached, there is no unitary generator able to disentangle the $A - B$ coupling.

10 Trajectories in the $\mathcal{H}$ space

In this Section I want to discuss the chaotic behavior, under certain conditions, of the trajectories in the $\mathcal{H}$ space. Let me start by recalling some of the features of the $SU(1,1)$ group structure (see, e.g., [33]).

$SU(1,1)$ realized on $C \times C$ consists of all unimodular 2×2 matrices leaving invariant the Hermitian form $|z_1|^2 - |z_2|^2$, $z_i \in C, i = 1, 2$. The complex z plane is foliated under the group action into three orbits: $X_+ = \{z : |z| < 1\}$, $X_- = \{z : |z| > 1\}$ and $X_0 = \{z : |z| = 1\}$.

The unit circle $X_+ = \{\zeta : |\zeta| < 1\}$, $\zeta \equiv e^{i\phi} \tanh\theta$, is isomorphic to the upper sheet of the hyperboloid which is the set **H** of pseudo-Euclidean bounded (unit norm) vectors $\mathbf{n} : \mathbf{n} \cdot \mathbf{n} = 1$. **H** is a Kählerian manifold with metrics

$$ds^2 = 4 \frac{\partial^2 F}{\partial\zeta \partial\bar{\zeta}} d\zeta \cdot d\bar{\zeta} \; , \tag{132}$$

and

$$F \equiv - \ln(1 - |\zeta|^2) \tag{133}$$

is the Kählerian potential. The metrics is invariant under the group action [33].

The Kählerian manifold **H** is known to have a symplectic structure. It may be thus considered as the phase space for the classical dynamics generated by the group action [33].

The $SU(1,1)$ generalized coherent states are recognized to be "points" in **H** and transitions among these points induced by the group action are therefore classical trajectories [33] in **H** (a similar situation occurs [33] in the $SU(2)$ (fermion) case).

Summarizing, the space of the unitarily inequivalent representations of the ccr, which is the space of the $SU(1,1)$ generalized coherent states, is a Kählerian manifold, $\mathcal{H} \equiv \{\mathcal{H}_\theta, \; \forall\theta\} \approx \mathbf{H}$; it has a symplectic structure and a classical dynamics is established on it by the $SU(1,1)$ action (generated by $\mathcal{G}$ or, equivalently, by p_θ: $\mathcal{H}_\theta \rightarrow \mathcal{H}_{\theta'}$). Variations of the θ–parameter induce transitions through the representations $\mathcal{H}_\theta = \{|0(\theta)\rangle\}$, i.e. through the physical phases of the system, the system order parameter being dependent on θ. These transitions are described as trajectories through the "points" in $\mathcal{H}$. One may then assume time-dependent θ: $\theta = \theta(t)$. For example, this is the case of dissipative systems and of non-equilibrium thermal field theories where $\theta_\kappa = \theta_\kappa(\beta(t))$, with $\beta(t) = \frac{1}{k_B T(t)}$.

It is interesting to observe that, considering the transitions $\mathcal{H}_\theta \rightarrow \mathcal{H}_{\theta'}$, i.e. $|0(\theta)\rangle \rightarrow |0(\theta')\rangle$, we have

$$\langle 0(\theta)|0(\theta')\rangle = e^{-\frac{V}{2(2\pi)^3} \int d^3\kappa F_\kappa(\theta,\theta')} \tag{134}$$

where $F_\kappa(\theta, \theta')$ is given by Eq. (133) with $|\zeta_\kappa|^2 = \tanh^2(\theta_\kappa - \theta'_\kappa)$, which shows the role played by the Kählerian potential in the motion over $\mathcal{H}$.

The result that the group action induces classical trajectories in $\mathcal{H}$ has been also obtained elsewhere [50, 51] on the ground of more phenomenological considerations.

With reference to the discussion presented in Sections 3 - 5, we may say that on the (classical) trajectories in $\mathcal{H}$ it is $x_+ = x_- = x_{classical}$, i.e. on these trajectories the quantum noise accounted for by y is fully shielded by the thermal bath (cf. Eq. (48)). In Section 5 (see [23]) it has been indeed observed that the y freedom contributes to the imaginary part of the action which becomes negligible in the classical regime, but is relevant for the quantum dynamics, namely in each of the "points" in $\mathcal{H}$ (i.e. in each of the spaces $\mathcal{H}_\theta$, for each θ) through which the trajectory goes as θ changes. Upon "freezing" the action of $G(\theta)$ (i.e. upon "freezing" the "motion" through the ui representations) the quantum features of $\mathcal{H}_\theta$, at given θ, become manifest. This relates to the 't Hooft picture [14] and to the results of refs. [15, 16] where dissipation loss in deterministic systems may manifest itself as quantum behavior (see Section 11).

Let me use the notation $|0(t)\rangle_\theta \equiv |0(\theta(t))\rangle$. For any $\theta(t) = \{\theta_\kappa(t), \forall\kappa\}$ it is

$$ {}_\theta\langle 0(t)|0(t)\rangle_\theta = 1 \ , \quad \forall t \ . \tag{135}$$

I will now restrict the discussion to the case in which, for any κ, $\theta_\kappa(t)$ is a growing function of time and

$$\theta(t) \neq \theta(t') \ , \quad \forall t \neq t', \quad \text{and} \quad \theta(t) \neq \theta'(t') \ , \quad \forall t, t' \ . \tag{136}$$

Under such conditions the trajectories in $\mathcal{H}$ satisfy the requirements for chaotic behavior in classical nonlinear dynamics. These requirements are the following [52]:

i) the trajectories are bounded and each trajectory does not intersect itself.

ii) trajectories specified by different initial conditions do not intersect.

iii) trajectories of different initial conditions are diverging trajectories.

Let $t_0 = 0$ be the initial time. The "initial condition" of the trajectory is then specified by the $\theta(0)$-set, $\theta(0) = \{\theta_\kappa(0), \forall\kappa\}$. One obtains

$$ {}_\theta\langle 0(t)|0(t')\rangle_\theta \underset{V\to\infty}{\longrightarrow} 0 \ , \quad \forall\, t\,, t' \ , \quad \text{with} \quad t \neq t' \ , \tag{137}$$

provided $\int d^3\kappa \ \ln\cosh(\theta_\kappa(t) - \theta_\kappa(t'))$ is finite and positive for any $t \neq t'$.

Eq. (137) expresses the unitary inequivalence of the states $|0(t)\rangle_\theta$ (and of the associated Hilbert spaces $\{|0(t)\rangle_\theta\}$) at different time values $t \neq t'$ in the infinite volume limit. The non-unitarity of time evolution, implied for example by the damping, is consistently recovered in the unitary inequivalence among representations $\{|0(t)\rangle_\theta\}$'s at different t's in the infinite volume limit.

The trajectories are bounded in the sense of Eq. (135), which shows that the "length" (the norm) of the "position vectors" (the state vectors at time t) in $\mathcal{H}$ is finite (and equal to one) for each t. Eq. (135) rests on the invariance of the Hermitian form $|z_1|^2 - |z_2|^2$, $z_i \in C, i = 1, 2$ and I also recall that the manifold of points representing the coherent states $|0(t)\rangle_\theta$ for any t is isomorphic to the product of circles of radius ${r_\kappa}^2 = \tanh^2(\theta_\kappa(t))$ for any κ.

Eq. (137) expresses the fact that the trajectory does not crosses itself as time evolves (it is not a periodic trajectory): the "points" $\{|0(t)\rangle_\theta\}$ and $\{|0(t')\rangle_\theta\}$ through which the trajectory goes, for any t and t', with $t \neq t'$, after the initial time $t_0 = 0$, never coincide. The requirement i) is thus satisfied.

In the infinite volume limit, we also have

$$ {}_\theta\langle 0(t)|0(t')\rangle_{\theta'} \underset{V\to\infty}{\longrightarrow} 0 \qquad \forall\, t\,, t' \quad , \quad \forall\, \theta \neq \theta' \ . \tag{138}$$

Under the assumption (136), Eq. (138) is true also for $t = t'$. The meaning of Eqs. (138) is that trajectories specified by different initial conditions $\theta(0) \neq \theta'(0)$ never cross each other. The requirement ii) is thus satisfied.

In order to study how the "distance" between trajectories in the space $\mathcal{H}$ behaves as time evolves, consider two trajectories of slightly different initial conditions, say $\theta'(0) = \theta(0) + \delta\theta$, with small $\delta\theta$. A difference between the states $|0(t)\rangle_\theta$ and $|0(t)\rangle_{\theta'}$ is the one between the respective expectation values of the number operator $A_\kappa^\dagger A_\kappa$. For any κ at any given t, it is

$$\Delta\mathcal{N}_{A_\kappa}(t) \equiv {\mathcal{N}'}_{A_\kappa}\big(\theta'(t)\big) - \mathcal{N}_{A_\kappa}\big(\theta(t)\big)$$

$$= {}_{\theta'}\langle 0(t)|A_\kappa^\dagger A_\kappa|0(t)\rangle_{\theta'} - {}_{\theta}\langle 0(t)|A_\kappa^\dagger A_\kappa|0(t)\rangle_{\theta} \tag{139}$$

$$= \sinh^2\big({\theta'}_\kappa(t)\big) - \sinh^2\big(\theta_\kappa(t)\big) = \sinh\big(2\theta_\kappa(t)\big)\delta\theta_\kappa(t) \ , \tag{140}$$

where $\delta\theta_\kappa(t) \equiv {\theta'}_\kappa(t) - \theta_\kappa(t)$ is assumed to be greater than zero, and the last equality holds for "small" $\delta\theta_\kappa(t)$ for any κ at any given t. By assuming that $\frac{\partial \delta\theta_\kappa}{\partial t}$ has negligible variations in time, the time-derivative gives

$$\frac{\partial}{\partial t}\Delta\mathcal{N}_{A_\kappa}(t) = 2\frac{\partial\theta_\kappa(t)}{\partial t}\cosh\big(2\theta_\kappa(t)\big)\delta\theta_\kappa \ . \tag{141}$$

This shows that, provided $\theta_\kappa(t)$ is a growing function of t, small variations in the initial conditions lead to growing in time $\Delta\mathcal{N}_{A_\kappa}(t)$, namely to diverging trajectories as time evolves.

In the assumed hypothesis, at enough large t the divergence is dominated by $\exp(2\theta_\kappa(t))$. For each κ, the quantity $2\theta_\kappa(t)$ could be thus thought to play the rôle similar to the one of the Lyapunov exponent.

Since $\sum_\kappa E_\kappa \dot{\mathcal{N}}_{A_\kappa} dt = \frac{1}{\beta} dS_A$, where E_κ is the energy of the mode A_κ and dS_A is the entropy variation associated to the modes A (cf. Eq. (42)) [25], the divergence of trajectories of different initial conditions may be expressed in terms of differences in the variations of the entropy (cf. Eqs. (139) and (141)):

$$\Delta \sum_\kappa E_\kappa \dot{\mathcal{N}}_{A_\kappa}(t) dt = \frac{1}{\beta}\big(dS'_A - dS_A\big) \ . \tag{142}$$

The discussion above thus shows that also the requirement iii) is satisfied. The conclusion is that trajectories in the $\mathcal{H}$ space exhibit, under the condition (136) and with $\theta(t)$ a growing function of time, properties typical of the chaotic behavior in classical nonlinear dynamics.

11 Deterministic dissipative systems and quantization

In Section 3 we have seen that the canonical quantization for the damped oscillator is obtained at the expense of introducing an "extra" coordinate y. The role of the "doubled" y coordinate is absolutely crucial in the quantum regime where it accounts for the quantum noise. When the classical solution $y = 0$ is adopted, the x system appears to be "incomplete"; the loss of information due to dissipation amounts to neglecting the bath and to the ignorance of the bath-system interaction, i.e. the ignorance of "where" and "how" energy flows out of the system. One can thus conclude that the loss of information occurring at the classical level due to dissipation manifests itself in terms of "quantum" noise effects arising from the imaginary part of the action, to which the y contribution is crucial. This result suggests to consider the approach to dissipation presented above in connection with the proposal put forward by 't Hooft in a series of papers [14]. He proposes that Quantum Mechanics may indeed result from a more fundamental deterministic theory as an effect of a process of information loss. He considers a class of deterministic Hamiltonian systems described by means of Hilbert space techniques. The quantum systems are obtained when constraints implementing the information loss are imposed on the original Hilbert space. The Hamiltonian for such systems is of the form

$$H = \sum_i p_i\, f_i(q)\,, \tag{143}$$

where $f_i(q)$ are non–singular functions of the canonical coordinates q_i. The equations for the q's (i.e. $\dot{q}_i = \{q_i, H\} = f_i(q)$) are decoupled from the conjugate momenta p_i and this implies [14] that the system can be described deterministically even when expressed in terms of operators acting on the Hilbert space. The condition for the deterministic description is the existence of a complete set of observables commuting at all times, called *beables* [53]. For the systems of Eq.(143), such a set is given by the $q_i(t)$ [14].

In order to cure the fact that the Hamiltonians of the type (143) are not bounded from below, one might split H in Eq.(143) as [14]:

$$H = H_I - H_{II} \quad , \quad H_I = \frac{1}{4\rho}\,(\rho + H)^2 \quad , \quad H_{II} = \frac{1}{4\rho}\,(\rho - H)^2 \;, \tag{144}$$

where ρ is a time–independent, positive function of q_i. H_I and H_{II} are then positively (semi)definite and $\{H_I, H_{II}\} = \{\rho, H\} = 0$. Then the constraint condition is imposed onto the Hilbert space:

$$H_{II}|\psi\rangle = 0\,, \tag{145}$$

which ensures that the Hamiltonian is bounded from below. This condition, indeed, projects out the states responsible for the negative part of the spectrum. In other words, one gets rid of the unstable trajectories [14].

In refs. [15] and [16] it has been shown that the system of damped-antidamped oscillators discussed in Section 3 does provide an explicit realization of 't Hooft mechanism. In addition, it has been also shown that there is a connection between the zero point energy of the quantum harmonic oscillator and the geometric phase of the (deterministic) system of damped/antidamped oscillators. This can be seen by noticing that the Hamiltonian Eq.(27) is of the type (143) with $i = 1, 2$ and with $f_1(q) = 2\Omega$, $f_2(q) = -2\Gamma$, provided one uses a set of canonical transformations which for brevity I do not report here (see [15]). By using $J_2 = -\frac{i}{2}(J_+ - J_-)$ and $\mathcal{C} = \frac{1}{2}(A^\dagger A - B^\dagger B)$ one may write Eq. (28) as

$$H = H_I - H_{II} \quad , \quad H_I = \frac{1}{2\Omega\mathcal{C}}(2\Omega\mathcal{C} - \Gamma J_2)^2 \quad , \quad H_{II} = \frac{\Gamma^2}{2\Omega\mathcal{C}} J_2^2\,. \tag{146}$$

Note that $\mathcal{C}$, being the Casimir operator, is a constant of motion, which ensures that once it has been chosen to be positive it will remain such at all times. The constraint (145) is now imposed by putting

$$J_2|\psi\rangle = 0\,, \tag{147}$$

and the physical states $|\psi\rangle$ are by this defined. It is now convenient to introduce

$$x_1 = \frac{x+y}{\sqrt{2}}\,, \quad x_2 = \frac{x-y}{\sqrt{2}}\,,$$

and

$$x_1 \;=\; r\cosh u\,, \;\; x_2 \;=\; r\sinh u\,, \tag{148}$$

in terms of which [27]

$$\mathcal{C} \ = \ \frac{1}{4\Omega m}\left[p_r^2 - \frac{1}{r^2}p_u^2 + m^2\Omega^2 r^2\right] \ , \quad J_2 \ = \ \frac{1}{2}\, p_u \, . \tag{149}$$

Of course, only nonzero r^2 should be taken into account in order for $\mathcal{C}$ to be invertible. If one does not use the operatorial formalism, then the constraint $p_u = 0$ implies $u = -\frac{\gamma}{2m}t$. Eq.(147) implies

$$H|\psi\rangle = H_I|\psi\rangle = 2\Omega\mathcal{C}|\psi\rangle = \left(\frac{1}{2m}p_r^2 + \frac{K}{2}r^2\right)|\psi\rangle \, , \tag{150}$$

where $K \equiv m\Omega^2$. H_I thus reduces to the Hamiltonian for the linear harmonic oscillator $\ddot{r} + \Omega^2 r = 0$. The physical states are even with respect to time-reversal ($|\psi(t)\rangle = |\psi(-t)\rangle$) and periodical with period $\tau = \frac{2\pi}{\Omega}$.

I will now introduce the states $|\psi(t)\rangle_H$ and $|\psi(t)\rangle_{H_I}$ satisfying the equations:

$$i\hbar\frac{d}{dt}|\psi(t)\rangle_H \ = H\,|\psi(t)\rangle_H \ , \tag{151}$$

$$i\hbar\frac{d}{dt}|\psi(t)\rangle_{H_I} \ = 2\Omega\mathcal{C}|\psi(t)\rangle_{H_I} \, . \tag{152}$$

Eq.(152) describes the two-dimensional "isotropic" (or "radial") harmonic oscillator. $H_I = 2\Omega\mathcal{C}$ has the spectrum $\mathcal{H}_I^n = \hbar\Omega n$, $n = 0, \pm1, \pm2, ...$ According to the choice for $\mathcal{C}$ to be positive, only positive values of n will be considered. The generic state $|\psi(t)\rangle_H$ can be written as

$$|\psi(t)\rangle_H = \hat{T}\left[\exp\left(\frac{i}{\hbar}\int_{t_0}^{t} 2\Gamma J_2 dt'\right)\right]|\psi(t)\rangle_{H_I} \ , \tag{153}$$

where $\hat{T}$ denotes time-ordering. Of course, here $\hbar$ is introduced on purely dimensional grounds and its actual value cannot be fixed by the present analysis.

One obtains [15]:

$${}_H\langle\psi(\tau)|\psi(0)\rangle_H = \ {}_{H_I}\langle\psi(0)|\exp\left(i\int_{C_{0\tau}} A(t')dt'\right)|\psi(0)\rangle_{H_I} \equiv e^{i\phi} \, , \tag{154}$$

where the contour $C_{0\tau}$ is the one going from $t' = 0$ to $t' = \tau$ and back and $A(t) \equiv \frac{\Gamma m}{\hbar}(\dot{x}_1 x_2 - \dot{x}_2 x_1)$. Note that $(\dot{x}_1 x_2 - \dot{x}_2 x_1)dt$ is the area element in the (x_1, x_2) plane enclosed by the trajectories (see Fig.4) (cf. Section 6). Notice also that the evolution (or dynamical) part of the phase does not enter in ϕ, as the integral in Eq.(154) picks up a purely geometric contribution [39].

Let me consider the periodic physical states $|\psi\rangle$. Following [39], one writes

$$|\psi(\tau)\rangle \ = e^{i\phi - \frac{i}{\hbar}\int_0^\tau \langle\psi(t)|H|\psi(t)\rangle dt}|\psi(0)\rangle \ = \ e^{-i2\pi n}|\psi(0)\rangle \, , \tag{155}$$

i.e. $\frac{\langle\psi(\tau)|H|\psi(\tau)\rangle}{\hbar}\tau - \phi = 2\pi n$, $n = 0, 1, 2, \ldots$, which by using $\tau = \frac{2\pi}{\Omega}$ and $\phi = \alpha\pi$, gives

$$\mathcal{H}_{I,eff}^n \equiv \langle\psi_n(\tau)|H|\psi_n(\tau)\rangle = \hbar\Omega\left(n + \frac{\alpha}{2}\right) \ . \tag{156}$$

The index n has been introduced to exhibit the n dependence of the state and the corresponding energy. $\mathcal{H}_{\mathrm{i},eff}^n$ gives the effective nth energy level of the physical system, i.e. the energy given by $\mathcal{H}_{\mathrm{i}}^n$ corrected by its interaction with the environment.

One thus see that the dissipation term J_2 of the Hamiltonian is actually responsible for the "zero point energy" $(n = 0)$: $E_0 = \frac{\hbar}{2}\Omega\alpha$.

I recall that the zero point energy is the "signature" of quantization since in Quantum Mechanics it is formally due to the non-zero commutator of the canonically conjugate q and p operators. Thus dissipation manifests itself as "quantization". In other words, E_0, which appears as the "quantum contribution" to the spectrum, signals the underlying dissipative dynamics. If one wants to match the Quantum Mechanics zero point energy, has to fix $\alpha = 1$, which gives [15] $\Omega = \frac{\gamma}{m}$.

In connection with the discussion presented in Section 3.1, the thermodynamical features of the dynamical rôle of J_2 can be revealed by rewriting Eq.(153) as

$$|\psi(t)\rangle_H = \hat{T}\left[\exp\left(i\frac{1}{\hbar}\int_{u(t_0)}^{u(t)} 2J_2 du'\right)\right]|\psi(t)\rangle_{H_I}\,, \tag{157}$$

where $u(t) = -\Gamma t$ has been used. Thus,

$$-\,i\hbar\frac{\partial}{\partial u}|\psi(t)\rangle_H = 2J_2|\psi(t)\rangle_H\,. \tag{158}$$

$2J_2$ appears then to be responsible for shifts (translations) in the u variable, as it has to be expected since $2J_2 = p_u$ (cf. Eq.(149)). One can write indeed: $p_u = -i\hbar\frac{\partial}{\partial u}$. Then, in full generality, Eq.(147) defines families of physical states, representing stable, periodic trajectories (cf. Eq.(150)). $2J_2$ implements transition from family to family, according to Eq.(158). Eq.(151) can be then rewritten as

$$i\hbar\frac{d}{dt}|\psi(t)\rangle_H = i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle_H + i\hbar\frac{du}{dt}\frac{\partial}{\partial u}|\psi(t)\rangle_H\,. \tag{159}$$

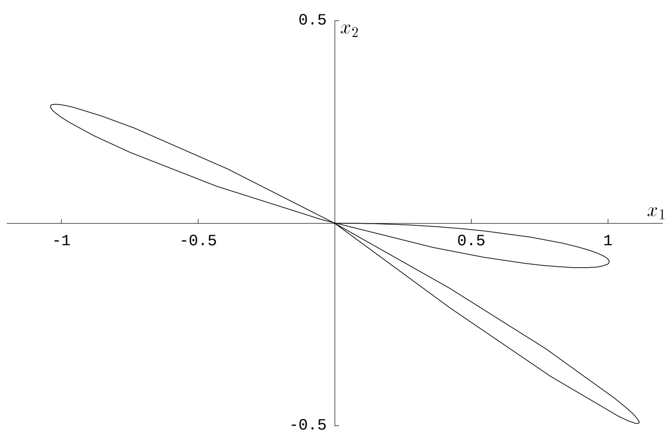

Fig. 4. Trajectories for $r_0 = 0$ and $v_0 = \Omega$, after three half-periods for $\kappa = 20$, $\gamma = 1.2$ and $m = 5$. The ratio $\int_0^{\tau/2}(\dot{x}_1 x_2 - \dot{x}_2 x_1)dt/\mathcal{E} = \pi\frac{\Gamma}{m\Omega^3}$ is preserved. $\mathcal{E}$ is the initial energy: $\mathcal{E} = \frac{1}{2}mv_0^2 + \frac{1}{2}m\Omega^2 r_0^2$.

The first term on the r.h.s. denotes of course derivative with respect to the explicit time dependence of the state. The dissipation contribution to the energy is thus described by the "translations" in the u variable. Now I consider the derivative

$$\frac{\partial S}{\partial U} = \frac{1}{T}\,. \tag{160}$$

From Eq.(146), by using $S \equiv \frac{2J_2}{\hbar}$ and $U \equiv 2\Omega\mathcal{C}$, one obtains $T = \hbar\Gamma$. Eq. (160) is the defining relation for temperature in thermodynamics (with $k_B = 1$) so that one could formally regard $\hbar\Gamma$ (which dimensionally is an energy) as the temperature, provided the dimensionless quantity S is identified with the entropy. In such a case, the "full Hamiltonian" Eq.(146) plays the role of the free energy $\mathcal{F}$: $H = 2\Omega\mathcal{C} - (\hbar\Gamma)\frac{2J_2}{\hbar} = U - TS = \mathcal{F}$. Thus $2\Gamma J_2$ represents the heat contribution in H (or $\mathcal{F}$). Of course, consistently, $\frac{\partial \mathcal{F}}{\partial T}\Big|_{\Omega} = -\frac{2J_2}{\hbar}$. In conclusion $\frac{2J_2}{\hbar}$ behaves as the entropy, which is not surprising since it controls the dissipative (thus irreversible) part of the dynamics. In this way the conclusions of Section 3 are reobtained. It is also suggestive that the temperature $\hbar\Gamma$ is actually given by the background zero point energy: $\hbar\Gamma = \frac{\hbar\Omega}{2}$.

Finally, I observe that

$$\frac{\partial \mathcal{F}}{\partial \Omega}\Big|_{T} = \frac{\partial U}{\partial \Omega}\Big|_{T} = mr^2\Omega\,, \tag{161}$$

which is the angular momentum, as expected since it is the conjugate variable of the angular velocity Ω.

The above results may suggest that the condition (147) can be then interpreted as a condition for an adiabatic physical system. $\frac{2J_2}{\hbar}$ might be viewed as an analogue of the Kolmogorov–Sinai entropy for chaotic dynamical systems.

Finally, I note that a reparametrization-invariant time technique in a specific model [54] also may lead to a quantum dynamics emerging from a deterministic classical evolution.

12 Conclusions

In this report I have reviewed some aspects of the algebraic structure of QFT related with the doubling of the degrees of freedom of the system under study. I have shown how such a doubling is related to the characterizing feature of QFT consisting in the existence of infinitely many unitarily inequivalent representations of the canonical (anti-)commutation relations and how this is described by the q-deformed Hopf algebra. I have considered several examples of systems and shown the analogies, or links, among them arising from the common algebraic structure of the q-deformed Hopf algebra.

I have considered the Wigner function and the density matrix formalism and shown that it requires the doubling of the degrees of freedom, which thus appears to be a basic formal feature also in Quantum Mechanics. In this connection I have considered the two-slit experiment and shown that quantum interference effects disappear in the limit of coincidence of the doubled variable $x_{\pm}$. Then I have shown how in QFT it is the q-deformed coproduct which is relevant and how Bogoliubov transformations are constructed in terms of it. I have considered quantum dissipation by studying the damped harmonic oscillator and the quantum Brownian motion and commented on how the arrow of time emerges from the intrinsic thermodynamic nature of dissipation. The vacumm structure is the one of the generalized coherent states. The connection (links) with the two-mode squeezed states and the noncommutative geometry in the plane emerges in a natural way in the discussion of these systems. In view of the similarity of some features of the coherent states with those of the fractals, it is an interesting question to ask whether fractal properties enter the QFT structure. A study on this point is in progress.

The relation with thermal field theory, in the thermo field dynamics formalism, reveals one further formal analogy with the systems mentioned above. In such a contest entropy appears to be a measure of the degree of entanglement between the system and the thermal bath in which it is embedded. This also relates with the connection between the doubled variables and quantum noise effects.

For brevity, here I have not considered the doubling of the degrees of freedom in expanding geometry problems (inflationary models) and in the quantization of the matter field in a curved background. For this I refer to the papers [17, 18, 19].

Finally, I have discussed how 't Hooft proposal, according to which the loss of information due to dissipation in a classical deterministic system manifests itself in the quantum features of the system, finds a possible description in the formal frame common to the systems mentioned above. In particular, I have shown that the quantum spectrum of the harmonic oscillator can be obtained from the dissipative character of the underlying deterministic system. In recent years, the problem of quantization of a classical theory has attracted much attention in gravitation theories and in non-hamiltonian dissipative system theories, also in relation with noncommutative space-time structures involving deformation theory (see for example [45]). By taking advantage of the fact that the manifold of the QFT unitarily inequivalent representations is a Kählerian manifold, I have shown that classical trajectories in such a manifold, which may exhibit chaotic behavior under some conditions, describe (phase) transitions among the inequivalent representations. The space of the QFT representations appears thus covered by a *classical blanket*.

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References

1. J. von Neumann: *Mathematical foundations of Quantum Mechanics*, (Princeton University Press, Princeton 1955)
2. H. Umezawa and G. Vitiello: *Quantum Mechanics*, (Bibliopolis, Napoli 1985)
3. O. Bratteli and D. W. Robinson: *Operator Algebras and Quantum Statistical Mechanics*, (Springer, Berlin 1979)
4. G.L. Sewell: *Quantum theory of collective phenomena*, (Clarendon Press, Oxford 1986)

5. H. Umezawa: *Advanced field theory: micro, macro and thermal concepts*, (AIP, N.Y. 1993)
U. Umezawa, M. Matsumoto and M. Tachiki: *Thermo Field Dynamics and Condensed States*, (North-Holland, Amsterdam 1982)
6. V. G. Drinfeld¿ In: *Proc. ICM Berkeley, CA*, ed. by A. M. Gleason, (AMS, Providence, R.I. 1986) p. 798
M. Jimbo, Int. J. Mod. Phys., A **4**, 3759 (1989)
Yu. I. Manin: *Quantum Groups and Non-Commutative Geometry*, (CRM, Montreal 1988)
7. E. Celeghini, T. D. Palev and M. Tarlini: Mod. Phys. Lett. B **5**, 187 (1991)
P. P. Kulish and N. Y. Reshetikhin: Lett. Math. Phys. **18**, 143 (1989)
V.Chari and A.Pressley: *A Guide to Quantum Groups*, (Cambridge University Press, Cambridge 1994).
8. Y. Takahashi and H. Umezawa: Collective Phenomena **2** 55 (1975); reprint in Int. J. Mod. Phys. **B10**, 1599 (1996)
9. E. Celeghini, S. De Martino, S. De Siena, A. Iorio, M. Rasetti and G. Vitiello: Phys. Lett. A **244**, 455 (1998)
S. De Martino, S. De Siena and G. Vitiello: Int. J. Mod. Phys. B **10**, 1615 (1996)
10. G. Vitiello: Int. J. Mod. Phys. B **18**, 785 (2004)
11. G. Vitiello, Braz. J. Phys. **35**, 351 (2005)
12. W.H.Zurek: Phys. Rev. D **24** 1516 (1981)
W.H.Zurek: Phys. Rev. D **26** 1862 (1982)
E.Joos and H.D.Zeh: Z. Phys. B **59** 223 (1985)
D.Giulini, E.Joss, C.Kiefer, J.Kupsch, I.O.Stamatescu and H.D.Zeh: *Decoherence and the appearance of a classical world in quantum theory* (Springer, Berlin, 1996)
13. C. Itzykson and J.B. Zuber: *Quantum Field Theory*, (McGraw-Hill Inc., 1980)
P.W. Anderson: *Basic Notions of Condensed Matter Physics*, (Menlo Park, Benjamin, 1984)
14. G. 't Hooft: In: *Basics and Highlights of Fundamental Physics*, (Erice 1999).
G. 't Hooft: Class. Quant. Grav. **16**, 3263 (1999)
G. 't Hooft: arXiv:hep-th/0105105
G. 't Hooft: Int. J. Theor. Phys., **42**, 355 (2003)
15. M. Blasone, P. Jizba and G. Vitiello: Phys. Lett., A **287**, 205 (2001)
16. M. Blasone, E. Celeghini, P. Jizba and G. Vitiello: Phys. Lett., A **310**, 393 (2003)
17. E. Alfinito, R. Manka and G. Vitiello: Class. Quant. Grav. **17**, 93 (2000)
E. Alfinito and G. Vitiello: Phys. Lett. A **252**, 5 (1999)
18. M. Martellini, P. Sodano and G. Vitiello: Nuovo Cim. A **48**, 341 (1978)
A. Iorio, G. Lambiase and G. Vitiello: Annals of Phys. **294**, 234 (2001)
19. A. Iorio, G. Lambiase and G. Vitiello: Annals Phys. **309**, 151 (2004)
20. J. Schwinger: J. Math. Phys. **2**, 407 (1961)
21. R. P. Feynman and F. L. Vernon: Annals of Phys., **24**, 118 (1963).
22. R. P. Feynman: *Statistical Mechanics*, (W. A. Benjamin Publ. Co., Reading, Ma. 1972)
23. Y. N. Srivastava, G. Vitiello and A. Widom: Annals Phys. **238**, 200 (1995)
24. M. Blasone, Y. N. Srivastava, G. Vitiello and A. Widom: Annals Phys. **267**, 61 (1998)

25. E. Celeghini, M. Rasetti and G. Vitiello: Annals Phys. **215**, 156 (1992)
26. H. Feshbach and Y. Tikochinsky: Trans. New York Acad. Sci., (Ser.II) **38**, 44 (1977)
27. M. Blasone, E. Graziano, O. K. Pashaev and G. Vitiello: Annals Phys., **252**, 115 (1996)
28. E. Celeghini, M. Rasetti, M. Tarlini and G. Vitiello: Mod. Phys. Lett. B **3**, 1213 (1989)
29. H. Bateman: Phys. Rev. **38**, 815 (1931)
P. M. Morse, H. Feshbach: *Methods of Theoretical Physics*, Vol.I, (McGraw–Hill, New York 1953) pag. 298
H. Dekker: Phys. Rept. **80**, 1 (1981)
M. Blasone and P. Jizba, Annals Phys. **312**, 354 (2004)
30. Y. Tsue, A. Kuriyama and M. Yamamura: Prog. Theor. Phys. **91**, 469 (1994)
S. Nishiyama, C. Providencia, J. da Providencia, Y. Tsue and M. Yamamura, Prog. Theor. Phys. **113**, 555 (2005)
31. V. E. Tarasov: Phys. Lett. A **288**, 173 (2001)
R. Banerjee and P. Mukherjee, J. Phys. A **35** (2002) 5591
R. Banerjee, Mod. Phys. Lett. A **17** (2002) 631
A. Mostafazadeh: arXiv:math-ph/0107001
R. S. Kaushal and H. J. Korsch: Phys. Lett. A **276**, 47 (2000)
32. J.R.Klauder and E.C.Sudarshan: *Fundamentals of Quantum Optics*, (Benjamin, New York 1968)
33. A. Perelomov: *Generalized Coherent States and Their Applications*, (Springer, Berlin 1986)
34. I.Ojima, *Ann. Phys. (N.Y.)* **137**, 1 (1981)
35. D. Cangemi, R. Jackiw and B. Zwiebach: Annals of Phys., **45**, 408 (1996)
36. S. De Filippo and G. Vitiello: Lett. Nuovo Cim. **19**, 92 (1977)
37. G. Vitiello: Int. J. Mod. Phys. B **9**, 973 (1995)
E. Alfinito and G. Vitiello: Int. J. Mod. Phys. B **14**, 853 (2000). [Erratum-ibid. B **14**, 1613 (2000)]
G.Vitiello: *My Double Unveiled - The dissipative quantum model of brain*, (John Benjamins Publ. Co., Philadelphia, Amsterdam 2001).
38. S. W. Hawking and R. Penrose: Sci. Am. **275**, 44 (1996)
39. J. Anandan, Y. Aharonov: Phys. Rev. Lett. **65**, 1697 (1990)
40. S. Sivasubramanian, Y. N. Srivastava, G. Vitiello, A. Widom: Phys. Lett., A **311**, 97 (2003)
41. H.P.Yuen: Phys. Rev. **A 13**, 2226 (1976)
42. A. Iorio and G. Vitiello: Mod. Phys. Lett. B **8**, 269 (1994)
A. Iorio and G. Vitiello: Annals Phys. **241**, 496 (1995)
43. E. Celeghini, M. Rasetti and G. Vitiello: Phys. Rev. Lett. **66**, 2056 (1991)
E. Celeghini, S. De Martino, S. De Siena, M. Rasetti and G. Vitiello: Mod. Phys. Lett. B **7**, 1321 (1993)
E. Celeghini, S. De Martino, S. De Siena, M. Rasetti and G. Vitiello: Annals Phys. **241**, 50 (1995)
44. G. V. Dunne, R. Jackiw and C. A. Trugenberger: Phys. Rev. D **41**, 661 (1990)
A. Iorio and T. Sykora: Int. J. Mod. Phys. A **17**, 2369 (2002)
A. Connes, M. R. Douglas and A. Schwarz: JHEP **9802**, 003 (1998)
N. Seiberg and E. Witten: JHEP **9909**, 032 (1999)
R. Banerjee and S. Ghosh: Phys. Lett. B **533**, 162 (2002)

45. P. Aschieri, C. Blohmann, M. Dimitrijevic, F. Meyer, P. Schupp and J. Wess: Class. Quant. Grav. **22**, 3511 (2005)
46. L. D. Landau, E. M. Lifshitz: *Quantum Mechanics*, (Pergamon Press, Oxford 1977) pp458 and 184
47. L. D. Faddeev, R. Jackiw: Phys. Rev. Lett. **60**, 1692 (1988)
48. H.Umezawa: In: *Banff/CAP Workshop on Thermal Field Theory*, eds. F.C.Khanna et al., (World Sci., Singapore 1994), p.109
49. L.C.Biedenharn: J.Phys. **A 22**, L873 (1989)
A.J.MacFarlane: J.Phys. A **22**, 4581 (1989)
50. R. Manka, J. Kuczynski and G. Vitiello: Nucl. Phys., B **276**, 533 (1986)
51. E. Del Giudice, R. Manka, M. Milani and G. Vitiello: Phys. Lett., A **206**, 661 (1988)
52. R. Hilborn, *Chaos and Nonlinear Dynamics*, (Oxford University Press, Oxford 1994)
53. J.S. Bell: *Speakable and Unspeakable in Quantum Mechanics*, (Cambridge University Press 1987)
54. H.T. Elze, O. Schipper: Phys. Rev. D **66**, 044020 (2002)