The Logic of Classes as Many: Lecture Nine
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1 Introduction

In lecture seven we formalized the simple logic of names that is an important part of the theory of reference in conceptual realism. The category of names, it will be remembered, includes both proper and common, and complex and simple, names, all of which occur as parts of quantifiers phrases. Quantifier phrases, of course, are what stand for the referential concepts of conceptual realism. We explained in that lecture how Leśniewski's logic of names, which Leśniewski called ontology, can be interpreted and reduced to our conceptualist logic of names, and how in that reduction we can explain and account for the oddities of Leśniewski's logic.

We concluded lecture seven with observations about the "nominalization," or transformation, of names as parts of quantifier phrases into objectual terms. What a "nominalized" name denotes as an objectual term, we said, is the extension of that name, i.e., of the concept that the name stands for in its role as part of a quantifier phrase. The extension of a name is not a set, nor a class as a "single object". Rather, the extension of a name is a class as many, i.e., a class as a plurality that is literally made up of its members. We listed three of the central features of classes as many as originally described by Bertrand Russell in his 1903 *Principles of Mathematics*. These are, first, that a vacuous name—that is, a name that names nothing—has no extension, which is not the same as having an empty class as its extension. In other words, there is no empty class as many. Secondly, the extension of a name that names just one thing is none other than that one thing; that is, a class as many that has just one member is identical with that one member. In other words it is because a class as many is literally many up of its members that it is nothing if it has no members, and why it is identical with its one member if it has just one member. Finally, that is also why a class as many that has more than one member is merely a plurality, or plural object, which is to say that as a plurality it is not a "single object," and therefore it cannot itself be a member of any class as many.

We begin here where left off in lecture seven, namely, with the logic of classes as many as an extension of the logic of names. We assume in this regard all of the axioms and theorems of the simple logic names given in that lecture. That logic consisted essentially of a free first-order logic of identity extended to include the category of names as parts of quantifiers, and where the quantifiers $\forall$ and $\exists$ can be indexed by name variables as well as objectual variables.

Now because names can be transformed into objectual terms we need a variable-binding operator that generates complex names the way that the λ-operator generates complex predicates.[1] We will use the cap-notation with brackets, $[\hat{x}A/...x...]$, for this purpose. Accordingly, where A is a name, proper or common, complex or simple, we take $[\hat{x}A]$ to be a complex name, but one in which the variable x is bound. Thus, where A is a name and φ is a formula, $[\hat{x}A]$, $[\hat{x}A/\varphi]$, and $[\hat{x}/\varphi]$ are names in which all of the free occurrences of x in A and φ are bound. We read these expressions as follows:

$[\hat{x}A]$ is read as 'the class (or group) of A',
$[\hat{x}A/\varphi]$ is read as 'the class (or group) of A that are φ', and
$[\hat{x}/\varphi]$ is read as 'the class (or group) of things that are φ'.

A formal language L is now understood as a set of predicate and name constants, instead of a set of predicates and objectual constants, as was originally described in our lecture on tense and modal logic. There will be objectual constants in a formal language as well, but they will be generated from the name constants by a "nominalizing" transformation. In our more comprehensive framework, which we are not concerned with here, objectual constants are also generated from predicate constants by the nominalizing transformation described in our fourth lecture. We extend the simultaneous inductive definition of names and formulas given in §3 of our previous lecture to include names of this complex form as well as follows: If L is a formal language, then:

- (1) Every name variable or name constant in L is a *name of* L;
- (2) if a, b are either objectual variables, name variables or name constants in L, or names of L the form $[\hat{x}B]$, where x is an objectual variable and B is a name (complex or simple) of L, then $(a = b)$ is a *formula* of L; and
- if φ, ψ are formulas of L, B is a name (complex or simple) of L, and x and C are an objectual and a name variable, respectively, then
(3) $\neg\varphi$,
(4) $(\varphi \rightarrow \psi)$,
(5) $(\forall x)\varphi$,
(6) $(\forall xB)\varphi$, and
(7) $(\forall C)\varphi$ are *formulas of* L, and
(8) B/φ,

[1] It should be remembered that in free logic being a substituend of free objectual variables—i.e., being an "*objectual term*"—is not the same as denoting a value of the bound objectual variables. That is, in free logic some objectual terms may denote nothing.

(9) $/\varphi$, and
(10) $[\hat{x}B]$ are *names* of L.

Note that by definition we now have formulas of the form $(\forall y[\hat{x}A])\varphi$, as well as those of the form $(\forall xA)\varphi$ and $(\forall yA(y/x))\varphi$ as in §3 of our previous lecture. We reduce the first to the last of these forms by adding the following axiom schema to those already listed in §3 of our previous lecture, but now understood to apply to our extended notions of name and formula:

Axiom 12: $(\forall y[\hat{x}A])\varphi \leftrightarrow (\forall yA(y/x))\varphi$, where y does not occur in A.

Because we are retaining the axioms and theorems of §3 of our previous lecture, our first axiom of the logic of classes as many, **Axiom 12**, begins where we left off, the last axiom of which was **axiom 11**, and the last theorem of which was **T7**.

We might note, incidentally, that **Axiom 12** is a conversion principle for complex names as parts of quantifier phrases. It is the analogue for complex names of the form $[\hat{x}/A]$ of λ-conversion for complex predicates of the form $[\lambda x\varphi]$.

Given our understanding of the existential quantifier as defined in terms of negation and the universal quantifier, this means we also have the following as a theorem (where y is free for x in A):

T8: $\vdash (\exists y[\hat{x}A])\varphi \leftrightarrow (\exists yA(y/x))\varphi$.

Two other axioms about the occurrence of names as objectual terms are:

Axiom 13: $(\exists A)(A = [\hat{x}B])$, where B is a name and A is a name variable that does not occur (free) in B; and

Axiom 14: $A = [\hat{x}A]$, where A is a simple name, i.e., a name variable or constant.

Axiom 13, incidentally, is a comprehension principle for complex names, and as such is the analogue for complex names of the comprehension principle **(CP^*_λ)** for complex predicates. What it says is that every complex name of the form $[\hat{x}B]$ is a value of the bound name variables, and therefore stands for a name, or nominal, concept. **Axiom 14** tells us that the name concept $[\hat{x}A]$ is none other than the name concept A.

It is noteworthy that our earlier **axiom 4** of §3 is now redundant and can be derived by Leibniz's law, **(LL^*),** from **axiom 13**. That is, by **(LL^*)**, $\vdash C = [\hat{x}B] \rightarrow [\varphi \rightarrow \varphi([\hat{x}B]/C)]$, and therefore by **(UG)**, **axioms 3, 5**, and tautologous transformations, $\vdash (\exists C)(C = [\hat{x}B]) \rightarrow [(\forall C)\varphi \rightarrow \varphi([\hat{x}B]/C)]$, and hence, by **axiom 13**,

T9: $\vdash (\forall C)\varphi \rightarrow \varphi([\hat{x}B]/C)$.

Strictly speaking, **T9** is actually slightly stronger than **axiom 4** in that it includes cases in which complex names occur as objectual terms, i.e., where some, or all, of the occurrences of C in φ may be as objectual terms, and hence where not all occurrences of $[\hat{x}B]$ in $\varphi([\hat{x}B]/C)$ can be replaced by B if B is a complex name of the form A/ψ or of the form $/\psi$. If C occurs in φ only as part of a quantifier phrase, then $\varphi([\hat{x}B]/C)$ is equivalent to $\varphi(B/C)$ by **axiom 12**.

We turn now to definitions of some of the concepts that are important in the logic of classes.

> Note that although we adopt the same symbols that are used in set theory to express membership, inclusion and proper inclusion, it should be kept in mind that the present notion of class is not that of set theory.

Definition 1 $x \in y \leftrightarrow (\exists A)[y = A \wedge (\exists zA)(x = z)]$.

Definition 2 $x \subseteq y \leftrightarrow (\forall z)[z \in x \rightarrow z \in y]$.

Definition 3 $x \subset y \leftrightarrow x \subseteq y \wedge y \nsubseteq x$.

> Note also that the argument for Russell's paradox for classes does not lead to a contradiction within this system as described so far, nor will it do with the axioms yet to be listed.

Rather, what it shows is that the Russell class as many does not "exist" in the sense of being the value of a bound objectual variable, which is not to say that *the name concept* of the Russell class does not have its own conceptual mode of being as a value of the bound name variables. Indeed, as the following definition indicates, the name, or nominal, concept of the Russell class can be defined in purely logical terms.

Definition 4 $Rus = [\hat{x}/(\exists A)(x = A \wedge x \notin A)]$.

That the Russell class as many does not "exist" as an object, i.e., as a value of the bound objectual variables, is important to note because it has been claimed that "the objective view" of plural objects, i.e., the view of them as objects (such as classes as many), is refuted by Russell's paradox.[2] The fact that the Russell class does not "exist" in the logic of classes as many is stated in the following theorem. (Proofs will be given only as footnotes.)

T10: $\vdash \neg(\exists x)(x = Rus)$.[3]

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[2] See, e.g., Schein 1993, pages 5, 15, and 32-37.

[3] **Proof.** By **axiom 13** and identity logic, $\vdash (\exists A)(Rus = A)$, and by definition 1, $\vdash Rus \in Rus \leftrightarrow (\exists A)[Rus = A \wedge (\exists xA)(x = Rus)]$, and therefore by Leibniz's law, a quantifier-confinement law and tautologous transformations, $\vdash Rus \in Rus \leftrightarrow (\exists xRus)(x = Rus)$. But then, by definition of Rus and **T8**, $\vdash (\exists xRus)(x = Rus) \leftrightarrow (\exists x/(\exists A)(x = A \wedge x \notin A))(x = Rus)$, and therefore, by **T1**, $\vdash (\exists xRus)(x = Rus) \leftrightarrow (\exists x)[(\exists A)(x = A \wedge x \notin A) \wedge x = Rus]$, from which, by Leibniz's law, it follows that $\vdash (\exists xRus)(x = Rus) \leftrightarrow (\exists x)[Rus \notin Rus \wedge x = Rus]$; and, accordingly, by quantifier-confinement laws, and tautologous transformations, $\vdash (\exists x)(x = Rus) \rightarrow (Rus \in Rus \leftrightarrow Rus \notin Rus)$, from which we conclude that $\vdash \neg(\exists x)(x = Rus)$. ■

What Russell's argument shows is that not every name concept has an extension that can be "object"-ified in the sense of being the value of a bound objectual variable.

Now the question arises as to whether or not we can specify a necessary and sufficient condition for when a name concept has an extension that can be "object"-ified, i.e., for when the extension of the concept can be proven to "exist" as the value of a bound objectual variable. In fact, the answer is affirmative. In other words, unlike the situation in set theory, such a condition can be specified for the notion of a class as many. An important part of this condition is Nelson Goodman's notion of an "atom," which, although it was intended for a strictly nominalistic framework, we can utilize for our purposes and define as follows.[4]

Definition 5 $Atom = [\hat{x}/\neg(\exists y)(y \subset x)]$.

This notion of an atom has nothing to do with physical atoms, of course. Rather, it corresponds in our present system approximately to the notion of an urelement, or "individual", in set theory. We say "approximately" because in our system atoms are identical with their singletons, and hence each atom will be a member of itself. This means that not only are ordinary physical objects atoms in this sense, but so are the propositions and intensional objects denoted by nominalized sentences and predicates in the fuller system of conceptual realism. Of course, the original meaning of 'atom' in ancient Greek philosophy was that of being indivisible, which is exactly what was meant by 'individual' in medieval Latin. An atom, or individual, in other words, is a "single" object, which is apropos in that objects in our ontology are either single or plural. We will henceforth use 'atom' and 'individual' in just this sense.

The following axiom (where y does not occur in A) specifies when and only when a name concept A has an extension that can be "object"-ified (as a value of the bound objectual variables).

Axiom 15: $(\exists y)(y = [\hat{x}A]) \leftrightarrow (\exists xA)(x = x) \wedge (\forall xA)(\exists zAtom)(x = z)$.

Stated informally, **axiom 15** says that the extension of a name concept A can be "object"-ified (as a value of the bound objectual variables) if, and only if, something is an A and every A is an atom.[5] An immediate consequence of this axiom, and of **T8** and **T1**, is the following theorem schema, which stipulates exactly when an arbitrary condition φx has an extension that can be "object"-ified.

T11: $\vdash (\exists y)(y = [\hat{x}/\varphi x]) \leftrightarrow (\exists x)\varphi x \wedge (\forall x/\varphi x)(\exists zAtom)(x = z)$.

Note that where φx is the impossible condition ($x \neq x$), it follows from **T11** that there can be no empty class, which, as already noted, is our first basic

[4] See Goodman 1956 for Goodman's account of atoms in nominalism.

[5] That something is an A is perspicuously symbolized by $(\exists y)(\exists xA)(y = x)$. But because $(\exists xA)(x = x) \leftrightarrow (\exists y)(\exists xA)(y = x)$ is provable, we will use $(\exists xA)(x = x)$ as a shorter way of saying the same thing.

feature of the notion of a class as many. We define the empty-class concept as follows and then note that its extension, by **T11**, cannot "exist" (as a value of the bound objectual variables), as well as that no *object* can belong to it.

Definition 6 $\Lambda = [\hat{x}/(x \neq x)]$.

T12a: $\vdash \neg(\exists x)(x = \Lambda)$.

T12b: $\vdash \neg(\exists x)(x \in \Lambda)$.

Finally, our last axiom concerns the second basic feature of classes as many; namely, that every atom, or individual, is identical with its singleton. In terms of a name concept A, the axiom stipulates that if at most one thing is an A and that whatever is an A is an atom, then whatever is an A is identical to the extension of A, which in that case is a singleton if in fact anything is an A. Where y does not occur in A, the axiom is as follows.

Axiom 16: $(\forall xA)(\forall yA)(x = y) \wedge (\forall xA)(\exists zAtom)(x = z) \rightarrow$
$(\forall yA)(y = [\hat{x}A])$.

A more explicit statement of the thesis that an atom is identical with its singleton is given in the following theorem.

T13: $\vdash (\exists zAtom)(x = z) \rightarrow x = [\hat{y}/(y = x)]$.[6]

By **T13**, it follows that every atom is identical with the extension of some name concept, e.g., the concept of being that atom. Of course, non-atoms, i.e., plural objects, are the extensions of name concepts as well (by the definitions of *Atom*, $\subset$, and $\in$), and hence anything whatsoever is the extension of a name concept.

T14: $\vdash (\exists zAtom)(x = z) \rightarrow (\exists A)(x = A)$.

T15: $\vdash \neg(\exists zAtom)(x = z) \rightarrow (\exists A)(x = A)$.

T16: $\vdash (\exists A)(x = A)$.

Note that if A is a proper name of an ordinary, physical object (and hence an atom), then, by the meaning postulate for proper names, the antecedent of **axiom 16** is true, and therefore, by **axioms 16** and **14**, $(\forall yA)(y = A)$. In other words, if A is a proper name of an atom, then $F(A) \leftrightarrow (\forall yA)F(y)$ is true, which in our conceptualist framework explains the role proper names have as "singular terms" (i.e., as substituends of free objectual variables) in free logic. That is, by Leibniz's law, **(UG)**, **axioms 2** and **6**, **T4**, a quantifier-confinement law,

$$(\forall yA)(y = A) \vdash F(A) \leftrightarrow (\forall yA)F(y).$$

[6] **Proof.** Where A be the nominal concept *thing-that-is-identical-to* x, i.e. $/(y = x)$, then, by **axiom 11** and **(LL*)**, $\vdash (\forall y/y = x)(\forall w/w = x)(y = w)$, and, similarly, $\vdash (\exists zAtom)(x = z) \rightarrow (\forall y/y = x)(\exists zAtom)(y = z)$. Therefore, by **axiom 16**, $\vdash (\exists zAtom)(x = z) \rightarrow (\forall y/y = x)(y = [\hat{y}/(y = x]))$. But, by **T6**, $\vdash (\exists zAtom)(x = z) \rightarrow (\exists z)(z = x)$, and therefore by **(∃/UI),** **T3** and **axiom 8**, $\vdash (\exists zAtom)(x = z) \rightarrow x = [\hat{y}/(y = x]$. ■

Of course, if A is a non-vacuous proper name of an ordinary object, then $(\exists yA)(y = A)$ is true as well, and hence $F(A) \leftrightarrow (\exists yA)F(y)$ as true as well. That is,

$$(\exists yA)(y = A) \vdash F(A) \leftrightarrow (\exists yA)F(y).$$

> What these last results indicate is that the role proper names have as "singular terms,"— i.e., as substituends of free objectual variables that purport to denote a "single" object— in standard free logic is reducible to, and fully explainable in terms of, the role proper names have in our logic of classes as many.

A consequence of **T13**, the definition of $\in$, **T8**, and Leibniz's law is the thesis that every atom is a member of itself. A similar argument, but without **T13**, shows that every object is a member of its singleton.

T17: $\vdash (\forall xAtom)(x \in x)$.

T18: $\vdash (\forall x)(x \in [\hat{z}/(z = x)])$.

Finally, we note that by definition of membership and Leibniz's law an object x belongs to the extension of a name concept A if, and only if, x is an A. From this it follows that only atoms can belong to an "object"-ified class as many, and hence that classes as many that are not atoms are not themselves members of any (real) classes as many, which is our third basic feature of classes as many.

T19: $\vdash x \in A \leftrightarrow (\exists yA)(x = y)$.

T20a: $\vdash (\forall x)[z \in x \rightarrow (\exists wAtom)(z = w)]$.

T20b: $\vdash \neg(\exists wAtom)(z = w) \rightarrow \neg(\exists x)(z \in x)$.[7]

2 Extensional Identity

The "nominalist's dictum," according to Nelson Goodman, is that "no two distinct things can have the same atoms."[8] Such a dictum, it would seem, should apply to classes as many as traditionally understood, regardless whether or not a more comprehensive framework containing such classes is nominalistic or not. In fact, the dictum is provable here if we assume an axiom of extensionality for classes.

But there is a problem with the axiom of extensionality. In particular, if the full unrestricted version of Leibniz's law is not modified, then having an axiom

[7] **Proof.** By definition of $\in$, $\vdash z \in x \rightarrow (\exists A)[x = A \wedge (\exists wA)(z = w)]$, and therefore, by **T6**, $\vdash z \in x \rightarrow (\exists w)(z = w)$. By **axiom 15**, $\vdash (\exists y)(A = y) \rightarrow (\forall zA)(\exists wAtom)(z = w)$; and therefore, by **axiom 10, T19**, and **(LL*)**, $\vdash (\exists y)(x = y) \wedge x = A \rightarrow (\forall z)[z \in A \rightarrow (\exists wAtom)(z = w)]$. But then, by quantifier-confinement laws, **T16**, **(LL*), (∃/UI)** and elementary transformations, $\vdash (\exists y)(x = y) \rightarrow [z \in x \rightarrow (\exists wAtom)(z = w)]$. Therefore, by **(UG)** and **axiom 7**, $\vdash (\forall x)[z \in x \rightarrow (\exists wAtom)(z = w)]$, which is **T20a.** **T20b** then follows by a quantifier-confinement law and tautologous transformations. ■

[8] Goodman 1956, p. 21.

of extensionality would seem to commit us to a strictly extensional framework even if it is not otherwise nominalistic. Name concepts that have the same extension at a given moment in a given possible world would then, by Leibniz's law, be necessarily equivalent, and therefore have the same extension at all times in every possible world, which is counter-intuitive. It is hoped, for example, that the extension of a common name concept such as 'country that is democratic' can have more and more members in it over time. Common name concepts of animals, e.g., 'buffalo', certainly have different extensions over time. Some, as in the case of names of plants and animals that have become extinct have changed their extensions radically from having millions of members to now having none. The idea that common name concepts cannot have different extensions over time, no less in different possible worlds, is a consequence we do not want in our broader framework of conceptual realism.

On the other hand, classes as many are extensional objects, and an axiom of extensionality that applies at least to classes as many is the only natural assumption to make in an ontology with classes as many, as in fact ours is. All objects, in other words, whether they are single or plural, are classes as many, and the idea that classes as many are not identical when they have the same members is difficult to reconcile with such an ontology.

Fortunately, there is an alternative, namely, that the full version of Leibniz's law as it applies to all contexts is to be restricted to atoms, i.e., individuals in the ontological sense. The restricted version for extensional contexts can then still be applied to pluralities, i.e., classes as many that have more than one member. Thus, in addition to the axiom of extensionality, we will take the following as an new axiom schema of our general framework.

Axiom 17: $(\exists z Atom)(x = z) \wedge (\exists z Atom)(y = z) \rightarrow [x = y \rightarrow (\varphi \leftrightarrow \psi)]$,
where ψ is obtained from φ by replacing one or more free occurrences of x by free occurrences of y.

This axiom is redundant if we do not add any nonextensional contexts, e.g., tense or modal operators, to the logic of classes as many. The reason is because, in a strictly extensional language, the full, unrestricted version of Leibniz's law is derivable from **axiom 9**.[9] In other words, **Axiom 9** remains in effect, but all we can prove from it is that Leibniz's law holds for all extensional contexts.[10]

> This distinction between how Leibniz's law applies to atoms and how it applies to classes as many in general is an ontological feature of our logic in that it distinguishes the individuality of atoms from the plurality of groups.

Indeed, unlike atoms, or individuals in the strict ontological sense, the identity of groups, or pluralities, i.e., classes as many with more than one member,

[9] As given in §3 of our previous lecture, **Axiom 9** is restricted to atomic formulas. The unrestricted version is then derivable by induction over the formulas of an extensional language. See, e.g., the proof given of **(LL)** in our second lecture.

[10] **Axiom 9** was Leibniz's law restricted to atomic formulas. The unrestricted version is derivable by induction over the formulas of the logic.

essentially reduces to the fact that they are made up of the same members, which does not justify the full, unrestricted ontological content of Leibniz's law.

A related point about **Axiom 17** is that it is an ontological thesis about the values of object variables and not about the objectual terms, e.g., name constants, that might be substituted for object variables. The validity of instantiating objectual terms for the object variables in this axiom depends on the contexts in which those variables occur and how "rigid" those objectual terms are with respect to those contexts. Proper names are assumed to be rigid with respect to tense and modal contexts, for example, but not in general with respect to belief and other cognitive modalities contexts except under special assumptions, such as knowing who, or what, the terms denote.

We now include the axiom of extensionality, which we will refer to hereafter as **(ext),** among the axioms.

Axiom 18 (ext): $(\forall z)[z \in x \leftrightarrow z \in y] \rightarrow x = y$

Goodman's nominalistic dictum that things are identical if they have the same atoms is now provable as the following theorem.

T21: $\vdash (\forall x)(\forall y)[(\forall z Atom)(z \in x \leftrightarrow z \in y) \rightarrow x = y]$.[11]

Note that by **T13** and the definition of $\in$, whatever belongs to an atom is identical with that atom, and therefore atoms are identical if, and only if, they the have the same members.

T22: $\vdash (\forall x Atom)[y \in x \rightarrow y = x]$.

T23: $\vdash (\forall x Atom)(\forall y Atom)[x = y \leftrightarrow (\forall z)(z \in x \leftrightarrow z \in y)]$.

Note also that by **T21** (and other theorems) it follows that everything "real", whether it is an atom or not, has an atom in it.
T24: $\vdash (\forall x)(\exists z Atom)(z \in x)$.[12]

Another useful theorem is the following, which, together with **T21**, shows that every non-atom must have at least two atoms as members. Of course, conversely, any "real" class (as many) that has at least two members cannot

[11] **Proof.** By **T5**, **T20a**, **(UG)**, quantifier-confinement laws, and elementary transformations, $\vdash (\forall x)[(\forall z Atom)(z \in x \leftrightarrow z \in y) \rightarrow (\forall z)(z \in x \rightarrow z \in y)]$, and similarly $\vdash (\forall y)[(\forall z Atom)(z \in x \leftrightarrow z \in y) \rightarrow (\forall z)(z \in y \rightarrow z \in x)]$, from which, given **(ext)**, **T21** follows. ■

[12] **Proof.** By **T5** and **T17**, $\vdash (\exists z Atom)(x = z) \rightarrow (\exists z Atom)(z \in x)$, and hence, by contraposition and the definition of *Atom*, $\vdash \neg(\exists z Atom)(z \in x) \rightarrow \neg(\exists z[\hat{x}\neg(\exists y)(y \subset x)])(x = z)$; and therefore, by **axioms 12, 11** and elementary transformations, $\vdash \neg(\exists z Atom)(z \in x) \rightarrow (\forall z)[x = z \rightarrow (\exists y)(y \subset z)]$, from which, by **(LL*)** and a quantifier-confinement law, it follows that $\vdash \neg(\exists z Atom)(z \in x) \rightarrow [(\exists z)(x = z) \rightarrow (\exists y)(y \subset x)]$; and therefore, by **(UG)** and **axioms 2** and **7**, $\vdash (\forall x)[\neg(\exists z Atom)(z \in x) \rightarrow (\exists y)(y \subset x)]$. Now, by definition of $\subset$, $\vdash \neg(\exists z Atom)(z \in x) \wedge y \subset x \rightarrow \neg(\exists z Atom)(z \in y)$, and therefore $\vdash \neg(\exists z Atom)(z \in x) \wedge y \subset x \rightarrow (\forall z Atom)[z \in x \leftrightarrow z \in y]$, and, accordingly by **(UG)** and **T21**, $\vdash (\forall x)(\forall y)[\neg(\exists z Atom)(z \in x) \wedge y \subset x \rightarrow x =_{ex} y]$. But then, by definition of $\subset$, $\vdash (\forall x)(\forall y)[\neg(\exists z Atom)(z \in x) \rightarrow (y \subset x \rightarrow x \subseteq y \wedge x \nsubseteq y)]$; and therefore, by quantifier logic, $\vdash (\forall x)[\neg(\exists z Atom)(z \in x) \rightarrow \neg(\exists y)(y \subset x)]$. Together with the above result, this shows that $\vdash (\forall x)[\neg(\exists z Atom)(z \in x) \rightarrow (\exists y)(y \subset x) \wedge \neg(\exists y)(y \subset x)]$, from which **T24** follows by quantifier logic. ■

be an atom, because then each of those members is properly contained in that class.

T25: $\vdash (\forall x)(\forall y)(y \subset x \to (\exists z Atom)[z \in x \wedge z \notin y])$.[13]

T26: $\vdash (\forall x)[\neg(\exists y Atom)(x = y) \leftrightarrow (\exists z_1/z_1 \in x)(\exists z_2/z_2 \in x)(z_1 \neq z_2)]$.[14]

Two consequences of the extensionality axiom, **(ext)**, are the strict identity of a class with the class of it members and the rewrite of bound variables for class expressions.

T27a: $\vdash x = [\hat{z}/(z \in x)]$.

T27b: $\vdash [\hat{x}A] = [\hat{y}A(y/x)]$, where y does not occur in A.[15]

3 The Universal Class

We have seen that, unlike the situation in set theory, the empty class as many does not "exist" (as a value of the bound objectual variables). But what about the universal class? In **ZF,** Zermelo-Fränkel set theory, there is no universal set, but in Quine's set theory **NF** (New Foundations) and the related set theory, **NFU** (New Foundations with Urelements), there is a universal set. In our present theory, the situation is more complicated. For example, if nothing exists, then of course the universal class does not exist. But, in addition, because something exists only if an atom does, i.e., by **T24** and **(∃/UI)**,

T28: $\vdash (\exists x)(x = x) \to (\exists x Atom)(x = x)$,

it follows that the universal class does not exist if there are no atoms, i.e., individuals—which is unlike the situation in set theory where classes exist whether or not there are any urelements, i.e., individuals. As it turns out, we can also show that the universal class does not exist if there are at least two atoms. If there is just one atom, however, the situation is more problematic.

First, let us define the universal class in the usual way, i.e., as the extension of the common name 'thing that is self-identical', and then note that whether or not

[13] **Proof.** By quantifier logic and definition of $\subset$, $\vdash y \subset x \to (\forall z Atom)(z \in y \to z \in x)$, and therefore, by **(UG)** and **T21**, $\vdash (\forall x)(\forall y)(y \subset x \to [(\forall z Atom)(z \in x \to z \in y) \to x =_{ex} y])$. But then, by definition of $\subset$ and $=_{ex}$, $\vdash (\forall x)(\forall y)(y \subset x \to [(\forall z Atom)(z \in x \to z \in y) \to x \subseteq y \wedge x \nsubseteq y])$, and hence $\vdash (\forall x)(\forall y)(y \subset x \to (\exists z Atom)[z \in x \wedge z \notin y])$. ■

[14] **Proof.** By **T25**, $\vdash (\forall x)(\forall y)(y \subset x \to (\exists z_1 Atom)[z_1 \in x \wedge z_1 \notin y])$, and by **T24** and **(∃/UI)**, $\vdash (\exists w)(y = w) \to (\exists z_2 Atom)(z_2 \in y)$. But, by **(LL*)** and definition of $\subset$, $\vdash y \subset x \wedge z_1 \notin y \wedge z_2 \in y \to z_2 \in x \wedge z_1 \neq z_2$, and therefore, by quantifier logic, $\vdash (\exists w)(y = w) \to (\forall x)[y \subset x \to (\exists z_1 Atom)(\exists z_2 Atom)(z_1 \neq z_2 \wedge z_1 \in x \wedge z_2 \in x)]$. Accordingly, by **(UG)**, **axiom 7, T1** and quantifier logic, $\vdash (\forall x)[(\exists y)(y \subset x) \to (\exists z_1 Atom/z_1 \in x)(\exists z_2 Atom/z_2 \in x))(z_1 \neq z_2)]$. But, by quantifier logic and definition of *Atom*, $\vdash (\forall x)[\neg(\exists y Atom)(x = y) \to (\exists y)(y \subset x)]$, from which the left-right-direction of **T26** follows. The converse direction is of course trivial for the reason already noted. ■

[15] **Proof.** By **(∃/UI)**, **T2**, and **(LL*)**, $\vdash (\exists y)(z = y) \to [z \in x \to (\exists y/y \in x)(z = y)]$, and therefore, by **T8** and **T19**, $\vdash (\exists y)(z = y) \to (z \in x \to z \in [\hat{z}/(z \in x)])$, and hence, by **axiom 7**, $\vdash (\forall z)(z \in x \to z \in [\hat{z}/(z \in x)])$. For the converse direction, by **T19**, **(LL*)**, and **T8**, $\vdash z \in [\hat{z}/(z \in x)] \to (\exists y/y \in x)(z = y)$; and hence $\vdash z \in [\hat{z}/(z \in x)] \to z \in x$. Therefore, by **(UG)**, $\vdash x = [\hat{z}/(z \in x)]$. The proof that $\vdash [\hat{x}A] =_{ex} [\hat{y}A(y/x)]$ follows from the definition of $\in$ and the rewrite rule for relative quantifiers, and **T27b** then follows by **(ext)**. ■

the name concept *thing-that-is-self-identical*, i.e., $[\hat{x}/(x = x)]$, can be "object"-ified (as a value of the bound objectual variables), nevertheless, everything "real" (in the sense of being the value of a bound objectual variable) is in it.

Definition 7 $\mathbf{V} = [\hat{x}/(x = x)]$.

T29: $\vdash (\forall x)(x \in \mathbf{V})$.[16]

> Note: all that T29 really says is that everything is a thing that is self-identical.

Now, by definition of $\in$, nothing can belong to the empty class, i.e., $x \notin \Lambda$, and therefore, by Leibniz's law, if anything at all exists, the universal class is not the empty class.

T30: $\vdash (\exists x)(x = x) \to \mathbf{V} \neq \Lambda$.

But it does not follow that the universal class "exists" if anything does. Indeed, as already noted above, we can show that if there are at least two atoms, then the universal class does not exist. First, let us note that if something exists (and hence, by **T28**, there is an atom), then the class of atoms exists, i.e., then the name concept *Atom* can be "object"-ified as a value of the bound objectual variables.

T31: $\vdash (\exists x)(x = x) \to (\exists y)(y = Atom)$.[17]

On the other hand, let us also note that

> if there are at least two atoms, then the class of atoms is not itself an atom.

T32: $\vdash (\exists x Atom)(\exists y Atom)(x \neq y) \to \neg(\exists z Atom)(z = Atom)$.[18]

By means of **T32**, we can now show that

> if there are at least two atoms, then the universal class does not "exist" (as a value of the objectual variables).

[16] **Proof.** By **axiom 8**, $\vdash (\forall x)(\exists y)(x = y) \leftrightarrow (\forall x)(\exists y)(y = y \wedge x = y)$, and therefore, by **T2**, $\vdash (\forall x)(\exists y)(x = y) \leftrightarrow (\forall x)(\exists y/y = y)(x = y)$, from which **T29** follows by **T8**, **T19** and the definition of V. ■

[17] **Proof.** By **axiom 15**, $\vdash (\exists x Atom)(x = x) \wedge (\forall x Atom)(\exists y Atom)(x = y) \to (\exists y)(y = Atom)$, from which, by **T28** and quantifier logic, **T31** follows. ■

[18] **Proof.** By definition of $\in$, **T8**, and elementary transformations, $\vdash x \neq y \to x \notin [\hat{z}/(z = y)] \wedge y \notin [\hat{z}/(z = x)]$, and therefore, by **T13** and **(LL*)**, $\vdash (\exists z Atom)(x = z) \wedge (\exists z Atom)(y = z) \wedge (x \neq y) \to x \notin y \wedge y \notin x$. By **T20a**, $\vdash (\exists z Atom)(x = z) \to x \subseteq Atom$, and, by **T19**, $\vdash (\exists z Atom)(y = z) \to y \in Atom$. Therefore, by definition of $\subset$, $\vdash (\exists z Atom)(x = z) \wedge (\exists z Atom)(y = z) \wedge y \notin x \to x \subset Atom$, and hence $\vdash (\exists z Atom)(x = z) \wedge (\exists z Atom)(y = z) \wedge (x \neq y) \to x \subset Atom$. But, by definition of *Atom*, $\vdash (\forall x)(\forall y)[x \subset y \to \neg(\exists z Atom)(z = y)]$, and hence, by **T31, T6**, and **(∃/UI)**, $\vdash (\exists z Atom)(x = z) \wedge x \subset Atom \to \neg(\exists z Atom)(z = Atom)$. Therefore, $\vdash (\exists x Atom)(\exists y Atom)(x \neq y) \to \neg(\exists z Atom)(z = Atom)$. ■

T33: $\vdash (\exists x Atom)(\exists y Atom)(x \neq y) \to \neg(\exists x)(x = \mathbf{V})$.[19]

Finally, in regard to the question of whether or not the universal class exists if the universe consists of just one atom, i.e., just one individual, note that if that were in fact the case, then, where A is a name of that one atom, the conjunction $(\exists z Atom)(z = A) \wedge (\forall z Atom)(z = A)$ would be true, and therefore the one atom A would be extensionally identical with the class of atoms, i.e., then, by **T31, T21**, and **(ext)**, $(A = Atom)$ would be true as well. Now, by **T29** and **T19**, $(\forall z Atom)[z \in Atom \leftrightarrow z \in \mathbf{V}]$ is provable, which, by **T21** might suggest that $(Atom = \mathbf{V})$ and hence $(A = \mathbf{V})$ are true as well. But in order for **T21** to apply in this case we need to know that $\mathbf{V}$ "exists," i.e., that $(\exists x)(x = \mathbf{V})$ is true. So, even if there were just one atom, we still could not conclude that the universal class is extensionally identical with that one atom.

4 Intersection, Union, and Complementation

Let us turn now to the Boolean operations of intersection, union and complementation for classes as many. We adopt the following standard definitions of each.

Definition 8 $x \cup y = [\hat{z}/z \in x \vee z \in y]$.

Definition 9 $x \cap y = [\hat{z}/(z \in x \wedge z \in y)]$.

Definition 10 $\bar{x} = [\hat{z}/z \notin x]$.

The following theorems regarding membership in the union and intersection of classes are consequences of **T19** and **T8**. The proof of the theorem regarding membership in the complement of a class is slightly more involved.

T34: $\vdash (\forall z)(z \in x \cup y \leftrightarrow z \in x \vee z \in y)$.

T35: $\vdash (\forall z)(z \in x \cap y \leftrightarrow z \in x \wedge z \in y)$.

T36: $\vdash (\forall z)(z \in \bar{x} \leftrightarrow z \notin x)$.[20]

Two immediate consequences of **T36** and **(ext)** (together with **T12b** and **T29**) are that the empty class is identical with the complement of the universal class, and that the universal class is identical with the complement of the empty class.

T37: $\vdash \Lambda = \bar{\mathbf{V}}$.

[19]**Proof.** Note that by **T20a** and **(∃/UI)**, $\vdash (\exists x)(x = V) \to (\forall x)[x \in V \to (\exists y Atom)(x = y)]$. But, by axiom **8, (UG)**, and **axioms 2** and **6**, $\vdash (\exists x)(x = V) \to (\exists x)(x = x)$, and hence, by **T31** and **(∃/UI)**, $\vdash (\exists x)(x = V) \to [Atom \in V \to (\exists y Atom)(y = Atom)]$. But, by **T31**, **T29**, and **(∃/UI)**, $\vdash (\exists x)(x = V) \to Atom \in V$, and hence, $\vdash (\exists x)(x = V) \to (\exists y Atom)(y = Atom)$. Accordingly, by **T32**, $\vdash (\exists x Atom)(\exists y Atom)(x \neq y) \to \neg(\exists x)(x = V)$. ■

[20]**Proof.** By definition of $\in$, $\vdash z \in \bar{x} \leftrightarrow (\exists A)[\bar{x} = A \wedge (\exists y A)(z = y)]$, and therefore, by **(LL*)** and **T8**, $\vdash z \in \bar{x} \to (\exists y/y \notin x)(z = y)$, and hence, by **T2** and **(LL*)**, $\vdash z \in \bar{x} \to z \notin x$. For the converse direction, note that by **T2** and **(LL*)**, $\vdash (\exists y)(z = y) \wedge z \notin x \to (\exists y/y \notin x)(z = y)$, and therefore, by the definitions of $\in$ and $\bar{x}$, $\vdash (\exists y)(z = y) \to [z \notin x \to z \in \bar{x}]$, and hence by **(UG) axioms 2** and **7**, and elementary logic, $\vdash (\forall z)(z \notin x \to z \in \bar{x})$. ■

T38: $\vdash \mathbf{V} = \bar{\Lambda}$.

In regard to the conditions for the existence of unions and intersections, we first prove a theorem that is useful in their respective proofs.

T39: $\vdash (\forall x)[(\exists z)(z \in x) \wedge (\forall z/z \in x)(\exists w Atom)(z = w)]$.[21]

T40: $\vdash (\forall x)(\forall y)(\exists z)(z = x \cup y)$.[22]

The related theorem for intersection requires a qualification, because some intersections—e.g., of distinct atoms—are empty, and, the empty class as many does not exist. Clearly, the relevant qualification is that the classes being intersected have a member in common.

T41: $\vdash (\forall x)(\forall y)[(\exists z)(z \in x \wedge z \in y) \rightarrow (\exists z)(z = x \cap y)]$.[23]

In regard to the existence of the complement of a class as many, we first note that if some atom is not in x, and therefore, by **T36**, is in $\bar{x}$, then the class as many of *atoms* in $\bar{x}$ exists, i.e., then $[\hat{z}Atom/(z \in \bar{x})]$ exists (as a value of the bound objectual variables). This result cannot be shown for $\bar{x}$ alone, however, because, e.g., where $x = \Lambda$, then, by **T38**, $\bar{x} = \mathbf{V}$, in which case $\bar{x}$ does not exist, or at least not if there exist two or more atoms. Also, in that case $[\hat{z}Atom/(z \in \bar{x})] = Atom$, and therefore, by **T28**, $[\hat{z}Atom/(z \in \bar{x})]$ exists even though $\bar{x}$ does not.

T42: $\vdash (\exists z Atom)(z \notin x) \rightarrow (\exists y)(y = [\hat{z}Atom/(z \in \bar{x})])$.[24]

Note that we can show that an atom is in $[\hat{z}Atom/(z \in \bar{x})]$ if, and only if, it is in $\bar{x}$, but we cannot use this result (**T43** below) to prove that $\bar{x}$ exists if $[\hat{z}Atom/(z \in \bar{x})]$ exists. In particular, we cannot use **T21** to prove $\bar{x} = [\hat{z}Atom/(z \in \bar{x})]$ unless we already know that both classes exist. The following theorems indicate what does in fact hold about the complement of a given class x.

T43: $\vdash (\forall z Atom)(z \in [\hat{z}Atom/z \notin x] \leftrightarrow z \in \bar{x})$.[25]

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[21] **Proof.** By **T6, (∃/UI), (LL*)**, and **T17**, $\vdash (\exists z Atom)(x = z) \rightarrow (\exists z)(z \in x)$, and, by **T26**, $\vdash (\forall x)[\neg(\exists z Atom)(x = z) \rightarrow (\exists z)(z \in x)]$; hence, $\vdash (\forall x)(\exists z)(z \in x)$. But then **T39** follows by **T20a** and quantifier logic. ■

[22] **Proof.** By **T39** (twice), $\vdash (\forall x)[(\exists z)(z \in x) \wedge (\forall z/z \in x)(\exists w Atom)(z = w)]$ and $\vdash (\forall y)[(\exists z)(z \in y) \wedge (\forall z/z \in y)(\exists w Atom)(z = w)]$, and therefore, by quantifier logic, $\vdash (\forall x)(\forall y)[(\exists z)(z \in x \vee z \in y) \wedge (\forall z/z \in x \vee z \in y)(\exists w Atom)(z = w)]$. Accordingly, by **T11**, $\vdash (\forall x)(\forall y)(\exists z_1)(z_1 = [\hat{z}/(z \in x \vee z \in y)])$, from which **T40** follows by definition of union. ■

[23] **Proof.** By **T39** (twice) and elementary logic, $\vdash (\forall x)(\forall y)(\forall z/z \in x \wedge z \in y)(\exists w Atom)(z = w)]$, and therefore, by **T11** and the definition of $\cap$, $\vdash (\forall x)(\forall y)[(\exists z/z \in x \wedge z \in y) \rightarrow (\exists z)(z = x \cap y)]$. ■

[24] **Proof.** By **axiom 15**, $\vdash (\exists z Atom)(z \in \bar{x}) \wedge (\forall z Atom/z \in \bar{x})(\exists w Atom)(z = w) \rightarrow (\exists y)(y = [\hat{z}Atom/(z \in \bar{x})])$; but, by **axiom 11** and quantifier logic, $\vdash (\forall z Atom/z \in \bar{x})(\exists w Atom)(z = w)$, and therefore, by **T36**, $\vdash (\exists z Atom)(z \notin x) \rightarrow (\exists y)(y = [\hat{z}Atom/(z \in \bar{x})])$. ■

[25] **Proof.** By **T19**, $\vdash (\exists y Atom/y \notin x)(z = y) \rightarrow z \in [\hat{y}Atom/y \notin x]$; and, by **T19** and **T36**, $\vdash (\forall z Atom)(z \in [\hat{y}Atom/y \notin x] \rightarrow z \in \bar{x})$. For the converse direction, by **T36** and **T2**, $\vdash (\forall z Atom)[z \in \bar{x} \rightarrow (\exists y Atom/y \notin x)(z = y)]$, and therefore, by **T19**, $\vdash (\forall z Atom)(z \in \bar{x} \rightarrow z \in [\hat{y}Atom/y \notin x])$. ■

T44: $\vdash (\exists y)(y = [\hat{z}Atom/z \notin x]) \wedge (\exists y)(y = \bar{x}) \rightarrow [\hat{z}Atom/z \notin x] = \bar{x}$.[26]

Finally, we should note that a set-theoretic semantics has been constructed for the logic of classes as many, and with respect to that semantics it has been shown that the logic is consistent.[27]

Metatheorem: The logic of classes as many as described here is consistent.

5 Leśniewskian Theses Revisited

As we explained in our previous lecture, Leśniewski's logic of names is reducible to our conceptualist logic of names. On our interpretation, the oddities of Leśniewski's logic are seen to be a result of his representing names, both proper and common, the way singular terms are represented in modern logic. The problem was not his view that proper and common names constitute together a syntactic category of their own, because that is how names are viewed in our conceptualist logic as well. But in our conceptualist logic proper and common names function as parts of quantifier phrases, i.e., expressions that stand for referential concepts in our analysis of the nexus of predication.

But if Leśniewski's logic of names is reducible to our conceptualist logic of names, then might not the oddities that arise in Leśniewski's logic also arise when names as parts of quantifier phrases are "nominalized" and occur as objectual terms in the logic of classes as many the way they occur in Leśniewski's logic of names? In other words, to what extent, if any, are there any theorems in our logic of classes as many that are counterparts of the theses of Leśniewski's logic that struck us as odd or noteworthy? Here, by a counterpart we mean a formula that results by replacing the names in a thesis of Leśniewski's logic by the "nominalized", or transformed, names of our logic of classes as many, and also, of course, replacing Leśniewski's epsilon 'ε' by our epsilon '$\in$'.

First, let us consider the validity of principle of existential generalization in Leśniewski's logic, i.e.,

$$\varphi(c/a) \rightarrow (\exists a)\varphi(a).$$

This principle is odd, we noted, when a is a vacuous name such as 'Pegasus', because in that case it follows from the fact that nothing is identical with Pegasus that something is identical with nothing, which is absurd. The counterpart of this thesis in our logic of classes as many is clearly invalid. For example, the empty class as many does not "exist" in our logic, and from that it does not follow that something exists that does not exist. Indeed, it is actually disprovable, as it should be. That is, the negation of

$$\neg(\exists x)(x = \Lambda) \rightarrow (\exists y)\neg(\exists x)(x = y)$$

is provable in our logic of classes as many.

[26] **Proof.** By **T43**, **T21**, **($\exists$/UI)**, and **(ext)**. ■

[27] See Cocchiarella 2002, Appendix 1.

Another thesis of Leśniewski's logic that is odd is the following:

$$(\exists a)(a \neq a).$$

Now, by **(UG)** and **axiom 8**, the negation of this thesis, namely $(\forall x)(x = x)$, is a theorem in our logic of classes as many. But of course stating the matter this way assumes that identity in Leśniewski's logic means identity *simpliciter*, which it doesn't. Identity is defined in Leśniewski's logic, in other words, and what the above thesis really means on Leśniewski's definition is the following:

$$(\exists a)\neg(a\ \varepsilon\ a).$$

Now the real counterpart of this thesis in our logic of classes as many is:

$$(\exists x)\neg(x \in x).$$

By quantifier negation, what this formula says is that not every object belongs to itself, which because all atoms belong to themselves, means that not every object is an atom. That is not a theorem of our logic, but it would be true if in fact there were at least two atoms, in which case there would then be a group, i.e., a class as many with more than one member, which, by definition, would not be an atom, and therefore, by T20b, not a member of anything, no less of itself. Thus, although the counterpart of the above Leśniewskian thesis is not a theorem, nevertheless it is not disprovable, and in fact it is true if there are at least two atoms, i.e., individuals in the ontological sense.

In regard to the Leśniewskian thesis,

$$a\ \varepsilon\ b \rightarrow a\ \varepsilon\ a.$$

we note first that the counterpart of this formula, namely,

$$z \in x \rightarrow z \in z,$$

is refutable if there is at least one plural object, i.e., one "real" object that is not an atom. This is because every "real" object is a member of the universal class (by **T29**), even though the universal class itself is not "real" if there are at least two atoms (**T33**). In other words, where z is a plural object, e.g., the class as many of citizens of Italy, then even though z is a member of the universal class, i.e., $z \in \mathbf{V}$, nevertheless $z \notin z$. That is, because z is a plural object, it is not an atom, and therefore (by **T20b**) z is not a member anything. Here, it should be kept in mind that even though $\mathbf{V}$ is not a value of the *bound* objectual variables, it is nevertheless a substituend of the *free* objectual variables. Hence, where z is a plural object, the following instance of the above formula,

$$z \in \mathbf{V} \rightarrow z \in z$$

is false.

There is a theorem that is somewhat similar to the above counterpart of Leśniewski's thesis, namely,

$$\vdash (\exists x)(z \in x) \rightarrow z \in z.$$

In other words, if z belongs to something "real", i.e., a value of the bound objectual variables, then z is an atom (by **T20a**) and therefore z belongs to itself (by **T17**). This theorem is similar to, but still not the same as, the Leśniewskian thesis.

Another theorem that is similar to, but not the same as, a thesis of Leśniewski's logic, is:

$$\vdash (\forall y)(\forall z)[x \in y \wedge y \in z \rightarrow x \in z].$$

This formula is provable because if x belongs to a "real" object y and y belongs to a "real" object z, then both x and y must be atoms (**by T20a**), in which case, $y = [\hat{w}/w = y]$ (by **T13**); and hence $x = y$ (because $x \in y$), and therefore $x \in z$ (because $y \in z$). This theorem is similar to the Leśniewskian thesis,

$$a \; \varepsilon \; b \wedge b \; \varepsilon \; c \rightarrow a \; \varepsilon \; c,$$

but, again, the strict counterpart of this Leśniewskian thesis, namely,

$$x \in y \wedge y \in z \rightarrow x \in z$$

is not provable in our logic, and is refutable if there are at least two atoms. Thus, if there are two "real" atoms a and b, then $y = [\hat{w}/(w = a \vee w = b)]$ is also "real" (by **axiom 15**). But then $y \in [\hat{w}/(w = y)]$ (by **T18**), and hence we would have $a \in y$ and $y \in [\hat{w}/(w = y)]$, and yet $a \notin [\hat{w}/(w = y)]$, because $a \neq b$, and hence $a \neq y$.

6 Groups and the Semantics of Plurals

One way in which the notion of a group is important is its use in determining the truth conditions of sentences that are irreducibly plural, i.e., sentences not logically equivalent to sentences that can be expressed without a plural reference to a group or plural predication about a group. An example of such a sentence is the so-called Geach-Kaplan sentence, 'Some critics admire only each other'.

Now the plural reference in this sentence is not just plural but irreducibly plural, and it cannot be logically analyzed by quantifying just over critics. The reference in this case is really to a group of critics, i.e., a class as many of critics having more than one member. The reference, moreover, is not to a *set* of critics, i.e., to an abstract object that is not itself a part of the physical world, but to a group of critics that is no less a part of the physical world than are the critics in the group. The difference between the group and its members is that the group, as a plural object, is ontologically founded upon its members as single objects.

The reference, moreover, is not to just any class as many of critics, and in particular not to any class as many that consists of just one member. A single critic who admires no one would in effect be a class as many of critics having exactly one member, and every member of this class would vacuously satisfies the condition that he admires only other members of the class. But it is counter-intuitive to claim that the sentence 'Some critics admire only each other' could be true only because there is a critic who admires no one. The sentence is true if, and only if, there is a group of critics every member of which admires only other members of the group.

In order to formulate this sentence properly we need first to define the notion of a group, and then note that, by definition, a group will have at least two members, and hence that a group is a plural object.

Definition 11 $Grp = [\hat{x}/(\exists y)(y \subset x)]$.

T45: $\vdash (\forall x Grp)(\exists z_1/z_1 \in x)(\exists z_2/z_2 \in x)(z_1 \neq z_2)$.

We can now represent the semantics of the sentence 'Some critics admire only each other' in terms of a group of critics instead of just a class as many of critics. This can be formulated as follows:

$$\text{[Some critics]}_{NP} \ \text{[admire only each other]}_{VP}$$

$$\swarrow \qquad\qquad\qquad\qquad \searrow$$

$$(\exists x Grp/x \subseteq [\hat{y}Critic]) \qquad\qquad (\forall y/y \in x)(\forall z)[Admire(y,z) \rightarrow z \in x \wedge z \neq y]$$

$$\searrow \qquad\qquad\qquad\qquad \swarrow$$

$$(\exists x Grp/x \subseteq [\hat{y}Critic])(\forall y/y \in x)(\forall z)[Admire(y,z) \rightarrow z \in x \wedge z \neq y].$$

Another example of an irreducibly plural reference is 'Some people are playing cards', where by 'some people' we do not mean that at least one person is playing cards, but that a group of people are playing cards, and that they are doing it together and not separately. The truth conditions of this sentence can be represented as follows where the argument of the predicate is irreducibly plural.

$$\begin{array}{ccc} & [\text{Some people}]_{NP}\ [\text{are playing cards}]_{VP} & \\ & \downarrow \qquad\qquad\qquad \downarrow & \\ & (\exists x Grp/x \subseteq [\hat{y}Person]) \qquad Playing\text{-}Cards(x) & \\ & \downarrow \qquad\qquad\qquad \downarrow & \\ & (\exists x Grp/x \subseteq [\hat{y}Person])Playing\text{-}Cards(x) & \end{array}$$

Of course, in saying that a group of people are playing cards we mean that each member of the group is playing cards, but also that the members of the group are playing cards *with* every other member of the group. That is,

$$(\forall x Grp/x \subseteq [\hat{y}Person])[Playing\text{-}Cards(x) \to (\forall y/y \in x)Playing\text{-}Cards(y)]$$

and

$$(\forall x Grp/x \subseteq [\hat{y}Person])[Playing\text{-}Cards(x) \to \\ (\forall y/y \in x)(\forall z/z \in x/z \neq y)Playing\text{-}Cards\text{-}with(y,z)]$$

are understood to be consequences of what is meant in saying that a group is playing cards. We understand the preposition '*with*' in this last formula, to be an operator that modifies a monadic predicate and generates a binary predicate by adding one new argument position to the predicate being modified. Thus, applying this modifier to 'x is playing cards' we get 'x is playing cards with y'. The added part, 'with y', represents a prepositional phrase of English.[28] The converse, however, does not follow in either case. That is, we could have every member of a group playing cards without the group playing cards together, and we could even have every member playing cards with every other member in separate games without all of them playing cards together in a single game.

Another type of referential expression that is irreducibly plural is the plural use of 'the', as in 'the inhabitants of Rome' and 'the Greeks who fought at Thermopylae'. On our reading these expressions are to be taken as referring to the inhabitants of Rome as a group and similarly to the Greeks who fought at Thermopylae as a group. In this way the plural use of 'the' can be reduced to the singular 'the', i.e., to a definite description of a group.

The singular 'the', as we described it in our fifth lecture, is represented by a quantifier (as are all determiners), in particular, $\exists_1$, where the truth conditions of an assertion of the form 'The A is F' are spelled out in essentially the Russellian manner (when the definite description is used with existential presupposition).

Consider now the sentence 'The Greeks who fought at Thermopylae are heroes', which we take to be equivalent to 'The group of Greeks who fought at Thermopylae are heroes'. Using $F(x)$ for the verb phrases 'x fought at Thermopylae', we can semantically represent the sentence as follows:

[28] Note that as described here 'with' cannot be iterated so as to result in a three-place predicate. We assume, in this regard, that 'playing cards with-with' is not grammatical or logically well-formed.

[The Greeks who fought at Thermopylae]$_{NP}$ [are heroes]$_{VP}$

↙ ↘

[The group of Greeks who fought at Thermopylae]$_{NP}$[are heroes]$_{VP}$

↓ ↓

$(\exists_1 xGrp/x = [\hat{x}Greek/F(x)])$ $(\forall y/y \in x)(\exists zHero)(y = z)$

↘ ↙

$$(\exists_1 xGrp/x = [\hat{x}Greek/F(x)])[\lambda x(\forall y/y \in x)(\exists zHero)(y = z)](x)$$

The truth conditions of this sentence amount to there being (now, at the time of the assertion) exactly one group of Greeks who fought at Thermopylae and every member of that group is a hero, which captures the intended content of the sentence in question. We might also note that another standard formulation of the English sentence, namely, that the class as many of Greeks who fought at Thermopylae is contained within the class as many of heroes,

$$[\hat{x}Greek/F(x)] \subseteq [\hat{x}Hero],$$

is a consequence of the above formulation; and, in fact, if it is assumed that $[\hat{x}Greek/F(x)]$ has at least two members and that each of its members is an atom, i.e., an individual, which in fact is the case, then the two formulations are equivalent to one another.

Another type of example is plural identity, as in:

Russell and Whitehead *are* the coauthors of PM.

Here, reference is to the group consisting of Russell and Whitehead, and what is predicated of this group is that it is identical with the group consisting of those who coauthored PM (*Principia Mathematica*). In other words, where 'A' and 'B' are name constants for 'Russell' and 'Whitehead', a plural subject of the form 'A and B' is analyzed as follows:

A and B

↓

The group consisting of A and B

↓

$$(\exists_1 xGrp/x = [\hat{z}/(z = A \vee z = B)$$

Similarly, the analysis of the phrase 'the coauthors of PM' is to be analyzed as follows:

the coauthors of PM

↓

The group of those who coauthored PM

↓

$$(\exists_1 yGrp/y = [\hat{z}/Coauthored(z, PM)])$$

The plural identity of the two groups can then be symbolized as,

$$(\exists_1 xGrp/x = [\hat{z}/(z = A \vee z = B)(\exists_1 yGrp/y = [\hat{z}/Coauthored(z, PM)])(x = y),$$

where it is the identity of two groups that is explicitly stated in the identity predicate. A similar analysis applies to the sentence

The triangles that have equal sides are the triangles that have equal angles.

That is, where 'A' is a name constant for 'triangle' and 'F' and 'G' are one-place predicates for 'has equal sides' and 'has equal angles', respectively, then the two plural definite descriptions can be represented as:

The triangles that have equal sides
↓
The group of triangles that have equal sides
↓
$(\exists_1 xGrp/x = [\hat{z}A/F(z)])$

and with a similar analysis for 'the group of triangles that have equal angles', the plural identity of the two groups can be symbolized as:

$$(\exists_1 xGrp/x = [\hat{z}A/F(z)])(\exists_1 yGrp/y = [\hat{z}A/G(x)])(x = y),$$

where, again, it is the identity of two groups that is stated in the identity predicate. We should note, however, that given the axiom of extensionality, this sentence is provably equivalent to

A triangle has equal sides if, and only if, it has equal angles.

which can be symbolized as:

$$(\forall xA)[F(x) \leftrightarrow G(x)].$$

In other words, strictly speaking, the truth conditions of this sentence does not involve an irreducibly plural reference to, or predication of, groups.

An example is an irreducibly plural predication is one where we predicate cardinal numbers of a group, as when we say that the Apostles *are* twelve. Here, the plural definite description, 'the Apostles' is understood to refer to the Apostles as a group, which means that we can symbolize the plural description as follows:

The Apostles
↓
The group of Apostles
↓
$(\exists_1 xGrp/x = [\hat{x}Apostle])$

What is predicated of this group is that it has twelve members. The verb phrase 'x has twelve members' can be symbolized as a complex predicate as follows,

$$x \text{ has twelve members}$$
$$\downarrow$$
$$[\lambda x(\exists^{12}y)(y \in x)](x)$$

As is well-known, the numerical quantifier $\exists^{12}$ is definable in first-order logic with identity, which we will not go into here. The important point is that this is really a plural predicate, i.e., it can be truthfully applied only to a plurality, namely a group with twelve members. The whole sentence can then be analyzed as follows:

$$[\text{The Apostles}]_{NP} \quad [\text{are twelve}]_{VP}$$
$$\swarrow \qquad\qquad\qquad \searrow$$
$$(\exists_1 x Grp/x = [\hat{x} Apostle]) \qquad\qquad [\lambda x(\exists^{12}y)(y \in x)](x)$$
$$\searrow \qquad\qquad\qquad \swarrow$$
$$(\exists_1 x Grp/x = [\hat{x} Apostle])[\lambda x(\exists^{12}y)(y \in x)](x)$$

or, by λ-conversion, more simply as

$$(\exists_1 x Grp/x = [\hat{x} Apostle])(\exists^{12}y)(y \in x).$$

7 The Cognitive Structure of Plural Reference and Predication

The logical analyses of plural reference and predication that we have described so far are primarily analyses of the truth conditions, i.e., the semantics, of plural reference and predication. They are not analyses of the cognitive structure of plural reference and predication as part of our speech and mental acts.

> The question is how can we account for the cognitive structure of plural reference and predication in terms of the logical forms that we use to represent our speech and mental acts.

What we propose is to formalize the pluralization of both common names and monadic predicates. We do this by means of an operator that when applied to a name results in the plural form of that name, and similarly when applied to a monadic predicate results in the plural form of that predicate. We will use the letter 'P' as the symbol for this plural operator and we will represent its application to a name A or predicate F by placing the letter 'P' as a superscript of the name or predicate, as in A^P and F^P.

Thus, we now extend the simultaneous inductive definition of the meaningful (well-formed) expressions of our conceptualist framework to include the following clauses:

1. if A is a name variable or constant, then A^P is a *plural name variable or constant*; respectively;

2. if A is a name, x is an object variable, and φx is a formula, then $[\hat{x}A/\varphi x]^P$ and $[\hat{x}/\varphi x]^P$ are *plural names*;

3. if $A/\varphi(x)$ is a (complex) name, then $(A/\varphi(x))^P = A^P/[\lambda x\varphi(x)]^P(x)$ and $[\hat{x}A/\varphi(x)]^P = [\hat{x}A^P/[\lambda x\varphi(x)]^P(x)]$;

4. if F is a one-place predicate variable or constant, or of the form $[\lambda x\varphi(x)]$, then F^P is a one-place *plural predicate*;

5. if A^P is a plural name, x is an object variable, and φ is a formula, then $(\forall xA^P)\varphi$ and $(\exists xA^P)\varphi$ are formulas.

In regard to clause (5), we read, e.g., '$(\forall xMan^P)$' as the plural phrase 'all men' and '$(\exists xMan^P)$' as the plural phrase 'some men', and similarly '$(\forall xDog^P)$' as 'all dogs' and '$(\exists xDog^P)$' as 'some dogs', etc. We note that a *plural* name is not a name *simpliciter* and that unlike the latter there is no rule for the "nominalization" of plural names, i.e., their transformation into objectual terms. This is because a nominalized name (occurring as an argument of a predicate) can already be read as plural if its extension is plural, and we do not want to confuse and identify a name *simpliciter* with its plural form.

Note also that only *monadic* predicates are pluralized. A two-place relation R can be pluralized in either its first- or second-argument position, or even both, by using a λ-abstract, as, e.g.,

$$[\lambda xR(x,y)]^P,$$
$$[\lambda yR(x,y)]^P,$$
$$[\lambda x[\lambda y[R(x,y)]^P(y)](x)]^P,$$

respectively; and a similar observation applies to n-place predicates for $n > 2$. Thus, for example, we can represent an assertion of 'Some people are playing cards with Sofia' by pluralizing the first-argument position of the two-place predicate 'x is playing cards with y' as follows:

$$\begin{array}{cc} [\text{Some people}]_{NP} & [\text{are playing cards with Sofia}]_{VP} \\ \downarrow & \downarrow \\ (\exists xPerson^P) & [\lambda xPlaying\text{-}Cards\text{-}with(x, Sofia)]^P(x) \end{array}$$

Semantically, of course, we understand the plural reference in this assertion to be to a group of people, a fact that is made explicit by assuming the following as a meaning postulate for all (nonplural) names A whether simple or complex:

$$(\exists xA^P)\varphi(x) \leftrightarrow (\exists xGrp/x \subseteq [\hat{y}A])\varphi(x). \qquad \textbf{(MPP1)}$$

Of course, if a group of people are playing cards with Sofia, then it follows that each person in the group is playing cards with Sofia, though, as already noted, the converse does not also hold. The one-direction implication from the plural to the singular can be described by assuming the following as part of the way that the monadic-predicate modifier '*with*' operates[29]:

$$(\forall x)([\lambda x Playing\text{-}Cards\text{-}with(x, Sofia)]^P(x) \rightarrow$$
$$(\forall y/y \in x)[\lambda x Playing\text{-}Cards\text{-}with(x, Sofia)](y)).$$

An example where the second argument of a relation is plural is a consequence of sentence 'Some men carry the piano downstairs', i.e., where the consequence is that each man in the group (qua individual) carries the piano downstairs with the other men in the group (qua group or plural object). First, where $F(x)$ is read as 'x carries the piano downstairs', we note that the sentence 'Some men carry the piano downstairs' can be analyzed as,

$$[\text{Some men}]_{NP} \; [\text{carry the piano downstairs}]_{VP}$$
$$\searrow \qquad\qquad \swarrow$$
$$(\exists x Man^P) \qquad\qquad F^P(x)$$
$$\searrow \qquad \swarrow$$
$$(\exists x Man^P)F^P(x)$$

which, by the above meaning postulate for plurals, **(MPP1)**, means that some group of men carry the piano downstairs. The consequence then is that some group of men is such that every man in the group carries the piano downstairs *with* the other men in the group. To analyze this, we need to represent what it means to refer to the men in the group other than a given man. For this we use the plural definite description, 'the men in x other than z', which can be symbolized as follows,

$$\text{the men in } x \text{ other than } z$$
$$\downarrow$$
$$(\exists_1 y(Man/(y \in x \wedge y \neq z))^P)$$

Finally, that there is a group of men such that every man in the group carries the piano downstairs with the other men in the group can now be represented as follows:

$$(\exists x Grp/x \subseteq [\hat{y}Man])(\forall z/z \in x)(\exists_1 y(Man/(y \in x \wedge y \neq z))^P)[\lambda y F\text{-}with(z, y)]^P(y),$$

which, by **(MPP1)** as applied to $(\exists_1 y(Man/(y \in x \wedge y \neq z))^P)$, reduces to

$$(\exists x Grp/x \subseteq [\hat{y}Man])(\forall z/z \in x)(\exists_1 y Grp/y = [\hat{y}Man/y \in x \wedge y \neq z])[\lambda y F\text{-}with(z, y)]^P(y),$$

where the relation 'z carries the piano downstairs with y' is taken as plural in its second-argument position.

[29] Note that because everything is a class as many, this condition applies even when x is an atom. In that case, of course, the condition is redundant.

Finally, let us turn to how the universal plural 'All A', the cognitive structure of which is represented by $(\forall x A^P)$, is to be semantically analyzed. Let us note first that if $(\forall x A^P)$ were taken as the logical dual of $(\exists x A^P)$, the way $(\forall x)$ is dual to $(\exists x)$, then the postulate for universal plural reference would be as follows:

$$(\forall x A^P)\varphi(x) \leftrightarrow (\forall x Grp/x \subseteq [\hat{y}A])\varphi(x). \qquad \text{(MPP?)}$$

Then, given that the cognitive structure of an assertion of 'All men are mortal' can be represented as,

$$(\forall x Man^P) Mortal^P(x),$$

it would follow that, semantically, the assertion amounts to predicating mortality to every group of men,

$$(\forall x Grp/x \subseteq [\hat{y}Man]) Mortal^P(x),$$

which is equivalent to saying without existential presupposition that the members of the entire group of men taken collectively *are* mortal:

$$(\forall x Grp/x = [\hat{y}Man]) Mortal^P(x).$$

This formula, given that the class as many of men is in fact a group—i.e., has more than one member—is in conceptualist terms very close to what Russell claimed in his 1903 *Principles*, namely, that the denoting phrase 'All men' in the sentence 'All men are mortal' denotes the class as many of men, which in fact happens to be a group.

But what if the class of men were to consist of exactly one man, as, e.g., at the time in the story of *Genesis* when Adam was first created. Presumably, the sentence 'All men are mortal' is true at the time in question. But is it a vacuous truth? In other words, is it true only because there is no group of men at that time but only a class as many of men having just one member?

Similarly, consider the sentence 'All moons of the earth are made of green cheese'. Presumably, this sentence is false and not vacuously true because there is no group of moons of earth but only a class as many with one member. In other words, regardless of the implicit duality of 'All A' with the plural 'Some A^P', we cannot accept the above rule **(MPP?)** as a meaning postulate for sentences of the form $(\forall x A^P)\varphi(x)$.

Yet, there is something to Russell's claim that the phrase 'All men' in the sentence 'All men are mortal' denotes the class as many of men and differs in this regard from what 'Every man' denotes in 'Every man is mortal'. In conceptualist terms, in other words, the referential concept that 'Every man' stands for is not the same as the referential concept that 'All men' stands for; nor is the predicable concept that 'is mortal' stands for the same as the predicable concept that 'are mortal' stands for.[30] A judgment that *all men are mortal* has a different cognitive structure from a judgement that *every man is mortal*,

[30] This difference in predicable concepts was missed by Russell and explains why his later rejection of classes as many as what 'All men' denotes was based on a confusion between singular and plural predication. See Russell 1903, p. 70.

even if semantically they have the same truth conditions. It is the difference in referential and predicable concepts—i.e., the difference between $(\forall x Man^P)$ and $(\forall x Man)$, on the one hand, and $Mortal^P(\ )$ and $Mortal(\ )$ on the other—that explains why the judgments are different.

Now the point of these observations is that instead of **(MPP?)** we can represent the difference between 'All A^p' and 'Every A' by adopting the following meaning postulate, which takes 'All A^P' to refer not just to every group of A but to every class as many of A, whereas 'Every A' refers to each and every A taken singly:

$$(\forall x A^P)\varphi(x) \leftrightarrow (\forall x/x \subseteq [\hat{y}A])\varphi(x). \qquad (\mathbf{MPP2})$$

Despite this difference, however, it follows that 'Every A is F' is logically equivalent to 'All A are F', i.e.,

$$(\forall x A)F(x) \leftrightarrow (\forall x A^p)F^p(x)$$

is valid in our conceptualist logic.

Let us note, incidentally, that the plural verb phrase 'are mortal', in symbols, $Mortal^P$, is semantically reducible to its singular form. That is, mortality can be predicated in the plural of a class as many if, and only if, every member of the class is mortal. In other words, as part of the meaning of the predicate 'mortal' we have the following as a meaning postulate:

$$Mortal^P(x) \leftrightarrow (\forall y/y \in x)Mortal(y).$$

It would be convenient, no doubt, if every plural predicate were reducible to its singular form the way $Mortal^P$ is, but that is not the case, as we noted earlier with the predicate $[\lambda x Playing\text{-}Cards\text{-}with(x, Sofia)]^P$, which is plural in its first argument position. Nor is it true of the complex predicate for carrying the piano downstairs with the other members of a group, which is plural in its second argument position.

The fact is that just as some references are irreducibly plural, so too some predications are irreducibly plural. Even though plural objects, i.e., groups, are ontologically founded upon the single objects that are their members, nevertheless plural objects are an irreducible part of the world as much as are single objects, i.e., individuals. What is needed for both our scientific and our commonsense frameworks is a logic that can account for plural objects and plural predication, whether in thought or in the world, no less so than it can account for single objects and singular predication. That is the logic we have presented here as a special part of the more general framework of conceptual realism.

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