# A Logic of the Possible and the Existent

# PIE System

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This paper presents a first investigation into the logic of the possible and the existent and develops an axiomatization based on two simple ideas: i) every possible individual can be considered as a *set of instant-objects*, connected by a particular relation, described in the analysis that follows as coincidence; ii) every instant-object can be constructed as a *complete set of attributes*, so that two instant-objects are different if and only if there are attributes that determine one of them and not the other[1]. The attributes that determine a possible instant-object are designated as essential attributes because they necessarily belong to that object. The property of existence is instead a different kind of attribute: when we say that a lamp exists, we do not intend to draw a distinction between this object and other possible objects – between this lamp and other possible lamps; actually, we are making a statement about the inclusion of the lamp in the actual world. Hence, to say that a lamp exists is to say that that lamp is actual or is part of the actual world. Existence is not an essential attribute; it does not determine the individual and so does not belong necessarily to any possible instant-object. The PIE system, presented below, seeks to give rigorous definition to the difference between essential attributes and existence, while describing at the same time the relation of coincidence connecting possible instant-objects. The paper consists of two parts: the first one deals with syntax; the second introduces a special kind of modal semantics and proves the correctness and completeness of the system.

## SYNTAX

The PIE system is an extension of the calculus PI of first-order predicates logic with identity. On the basis of this extension it is possible to deal, from the logical point of view, with the existential predicates, i.e. the predicate of existence and its modalizations (non real predicates in Kantian terminology) and the essential or non-existential predicates (real predicates in Kantian terminology). The modal system used as the basis of the treatment is S5, the system of unconditioned necessity.

[1] Likewise, in possible worlds semantics, a possible world conceived as a history can be constructed as a sequence of instant-worlds defined as complete sets of states of affairs.

## 1. Extension of the language

The set of logical signs comprises the signs for the existence predicate E, the coincidence relation ≈, and the necessity operator ❑. The set of well formed formulas is extended in the obvious way. Moreover, the possibility operator is introduced by the standard definition: $\Diamond\alpha$ := def. $\neg$❑$\neg\alpha$.

## 2. Extension of the calculus

*2.1. Primitive rules*

Necessitation rule N (by ❑(X) is meant the set of the necessitations of all the formulae included in X):

$$\frac{X \vdash \alpha}{\Box(X) \vdash \Box\alpha}$$

Axiom T: $\vdash \Box\alpha \rightarrow \alpha$
Axiom 5: $\vdash \Diamond\Box\alpha \rightarrow \Box\alpha$
Axiom NE: $\alpha \vdash \Box\alpha$ if E does not occur in $\alpha$

Axiom NE can be called the principle of necessitation of the essential properties: if individual x is determined by the property expressed by the formula $\alpha$, and if that property is indifferent with respect to existence, because the existence predicate does not occur in it, then the property expressed by $\alpha$ necessarily belongs to individual x. In other words, it is an essential property which, although it says nothing about the existence of the individual of which it is predicated, necessarily pertains to that individual as possible. The axiom obviously embodies and gives rigour to the Kantian distinction between essence and existence.

Axiom NI: $\vdash \Box\exists xEx$

Axiom NI states the impossibility of nothing: a world in which nothing exists is impossible. The plausibility of NI will be discussed below.

Axioms on the coincidence relation ≈

1.Symmetry of ≈: $x \approx y \vdash y \approx x$
2.Transitivity of ≈: $x \approx y \wedge y \approx z \vdash x \approx z$

3. Rule =≈: $x{=}y \vdash x{\approx}y$
4. Rule ≈=: $Ex \wedge Ey \wedge x{\approx}y \vdash x{=}y$

From axioms 1-3 it follows that ≈ is an equivalence relation. Detailed illustration of the meaning of these axioms will be given below. For the moment suffice it to say that they concern the identity relation between possibles indicated by ≈ as distinct from the usual identity relation as indicated by =.

*2.2. Usual derivable rules*

T1. a) $\Box(\alpha \wedge \beta) \vdash \Box\alpha \wedge \Box\beta$ b) $\Box\alpha \wedge \Box\beta \vdash \Box(\alpha \wedge \beta)$
T2. a) $\Box\alpha \vdash \neg\Diamond\neg\alpha$ b) $\Diamond\alpha \vdash \neg\Box\neg\alpha$
T3. a) $\Box\Diamond\alpha \vdash \neg\Diamond\Box\neg\alpha$ b) $\Diamond\Box\alpha \vdash \neg\Box\Diamond\neg\alpha$
T4. $\alpha \vdash \beta \Rightarrow \Diamond\alpha \vdash \Diamond\beta$
T5. $\alpha \vdash \Box\Diamond\alpha$
T6. $\Diamond\Box\alpha \vdash \alpha$
T7. $\Diamond\alpha \vdash \beta \Rightarrow \alpha \vdash \Box\beta$
T8. Barcan formula 1 (BF1): $\forall x\Box\alpha \vdash \Box\forall x\alpha$
T9. Barcan formula 2 (BF2): $\Diamond\exists x\alpha \vdash \exists x\Diamond\alpha$
T10. $\Box\forall x\alpha \vdash \forall x\Box\alpha$
T11. $\exists x\Diamond\alpha \vdash \Diamond\exists x\alpha$
T12. $\Diamond\forall x\alpha \vdash \forall x\Diamond\alpha$
T13. $\exists x\Box\alpha \vdash \Box\exists x\alpha$

Note that $\forall x\Diamond\alpha \vdash \Diamond\forall x\alpha$ and $\Box\exists x\alpha \vdash \exists x\Box\alpha$ do not hold in general. In short:

| | |
|---|---|
| BF1 | $\forall x\Box\alpha \vdash \Box\forall x\alpha$ |
| BF2 | $\Diamond\exists x\alpha \vdash \exists x\Diamond\alpha$ |
| T10 | $\Box\forall x\alpha \vdash \forall x\Box\alpha$ |
| T11 | $\exists x\Diamond\alpha \vdash \Diamond\exists x\alpha$ |
| T12 | $\Diamond\forall x\alpha \vdash \forall x\Diamond\alpha$ |
| T13 | $\exists x\Box\alpha \vdash \Box\exists x\alpha$ |
| | $\forall x\Diamond\alpha \nvdash \Diamond\forall x\alpha$ |
| | $\Box\exists x\alpha \nvdash \exists x\Box\alpha$ |

T14. $x{=}y \vdash \Box(x{=}y)$
Note that the rule can also be obtained by means of NE.

T15. $x \neq y \vdash \Box(x \neq y)$
In this case, too, the rule can be directly obtained from NE.

T16. $\exists x\alpha(x) \vdash \exists x\Diamond\alpha(x) \vdash \Diamond\exists x\alpha(x)$

Derivation:

| | |
|---|---|
| $\alpha(x) \vdash \Diamond\alpha(x)$ | by Axiom T |
| $\alpha(x) \vdash \exists x\Diamond\alpha(x)$ | I$\exists$ |
| $\exists x\alpha(x) \vdash \exists x\Diamond\alpha(x)$ | $\exists$I |
| $\exists x\alpha(x) \vdash \Diamond\exists x\alpha(x)$ | by T11 |

T17. $\Diamond\exists x\alpha(x) \vdash \exists x\alpha(x) \vdash \exists x\Diamond\alpha(x)$ (on the condition that E does not occur in $\alpha$)

Derivation:

| | |
|---|---|
| $\exists x\alpha(x) \vdash \Box\exists x\alpha(x)$ | NE |
| $\Diamond\exists x\alpha(x) \vdash \Diamond\Box\exists x\alpha(x)$ | by T4 |
| $\Diamond\exists x\alpha(x) \vdash \exists x\alpha(x)$ | by T6 |
| $\Diamond\exists x\alpha(x) \vdash \exists x\Diamond\alpha(x)$ | by T16 |

## SEMANTICS

The central concept of PIE semantics is that of *universal model*, which is based in its turn on the concept of *universal modal frame*. First presented here, therefore, is the concept of universal frame followed by that of model. Then, after enunciation of the lemmas of coincidence and conversion extended to the new semantics, the PIE theorems of correctness and completeness will be discussed.

### 1.1. Universal modal structure

A universal modal frame, or a universal frame *simpliciter*, F, is a set of possible worlds correlated and determined on the basis of the objects existing in them. In formal terms:

$$F = \langle W, R, \boldsymbol{U}, \boldsymbol{E} \rangle$$

Where:

- W is a non empty set of possible worlds
- R := W x W is a total relation on W
- ***U*** := <U,***P***>
- ***E*** := W $\mapsto$ $\wp$(U) is a function from possible worlds to the set of subsets of U

<U,***P***> is a pair constituted by a set U of possible objects characterized by attributes, properties and relations taken from the set ***P***

*1.1. Possible worlds and the accessibility relation*: W and R

W is a set of *ontologically possible worlds*. By “ontologically possible” world we mean a maximal set of states of affairs constituting a *real alternative*, not a purely *logical* one, to the maximal set of the states of affairs that make up the actual world. We assume that there is a difference in principle between the purely logical or analitical possibility of a world and its real possibility. The importance of ontological or real possibility compared to actuality – i.e. its non-identification with actuality – follows from the fact that otherwise the category of possibility would be entirely trivialized. In effect, analytical possibility resides in the pure consistency of such worlds, whereas their real possibility requires, besides consistency, also a foundation for their possibility; that is, they must be founded on something *real*. Now, for a non-determinist physicist, the states of the world connected with the different possible paths of its development are founded on the *real* structure of the world. They are therefore real alternatives to the actual world. Real alternatives to the actual world are the possible worlds that, according to a proponent of freedom, are within the capacity of a free man to produce. Real alternatives to the actual world are also the different states that the world has *really* assumed and will assume in the course of its history, according to the one-way conception of a determinist physicist: in this case, too, although the possibilities relative to the past have now gone and those of the future are still to come, the past states of the world have nevertheless been a real alternative to the actual world, and its future states will be so in entirely analogous manner. Even in the light of only these examples, the modality of real possibility differs sharply from that of analytical possibility, and at the same time is not at risk of being collapsed into that of actuality.[2] The conception – for example *à la* Hartmann –

[2] A clear distinction between logical possibility and physical possibility, a kind of our ontological possibility, is drawn in McCall, *A Model of the Universe*, Oxford 1994, p. 8: «What is logically possible, unlike what is physical possible, depends in no way upon what other events or states of affairs obtain, with the exception of those that logically imply it or logically exclude it. But there is no logical implication or logical exclusion across times, meaning that no state of affairs obtaining at one time either implies or excludes any state of affairs

according to which real possibility coincides with actuality is a extreme case of interpretation given to the exigency of providing an ontological foundation for the possible, and it takes concrete form in trivialization of the notion itself of possibility. In short, W includes all those worlds that differ in the number or attributes of the individuals actualized in them. Two worlds are therefore different if they comprise at least one actual individual characterized differently: the world in which this lamp is switched on is different from the world in which this same lamp is switched off, or determined according to all the properties that it possesses in the world wherein it is switched on apart from the property of being switched on.

The relation R of accessibility between worlds is a total relation. It establishes that every world is a possible alternative to any other. Expressed in relation R is the unconditioned nature of the notion of ontological possibility conveyed by the structure being presented here. This does not conflict with the idea defended above of real possibility; on the contrary, it is its consequential development. Forcefully asserted above, in fact, was the distinction between analytical possibility and real possibility, and we argued in favour of the non-emptiness of the notion of real possibility, providing examples of which two pertained to the sphere of physics. One might thus have gained the impression that the real possibility being discussed was physical possibility. This is to some extent true, but our only purpose in providing these examples was to show the meaningfulness of the notion of real possibility as opposed to analytical possibility. Having obtained this result, it is entirely reasonable to detach the notion of possibility from its physical connotation, but preserving its connotation of real possibility so that it can be semanticized in an idealized context. The resulting notion is thus that of ontological real possibility which serves the purpose of our system.[3]

### *1.2. Universal structure:* ***U***

A universal structure ***U*** consists of a set U of possible objects characterized by attributes (properties and relations) taken from the set ***P***:

obtaining at any another time. Hence what is logically possible at any time is independent of what states of affairs obtain at other times, while what is physically possibile is not. The physical modalities, unlike the logical modalities, are relative». Yet ontological possibility, unlike physical possibility, is not relative to initial conditions: what had obtained, given certain conditions, is no longer possible, from the physical point of view, given different conditions, but is still ontologically possible, according to the classical principle *ab esse ad posse*.

[3] Ontological possibility is then interpreted as metaphysical possibility, because cannot be completely reduced to the physical one.

$$\boldsymbol{U} = \langle \mathrm{U}, \boldsymbol{P} \rangle$$

By ‘possible (individual) object’ is meant an analytically possible (individual) object, i.e. an (individual) object which necessarily satisfies the sole requirement of coherence. Accordingly, analytically (individual) objects are all thinkable (individual) objects. The set of analytically possible individuals is divided in its turn into two distinct sub-sets: the sub-set of really possible individuals and the sub-set of purely possible (purely thinkable or hypothetical) ones. The former are individuals actualized (i.e. endowed with existence) in at least one possible world, while the latter are individuals not actualized (i.e. not endowed with existence) in any possible world. In other words, being an analytically possible individual, or one thinkable in the strict sense, coincides with being an individual different from nothing, and it is predicated of both whatever can be actual within a world and of whatever, although it cannot be actual, can be thought or hypothesised as actual.

Schematically:

| analytically possible individuals: $U = U_1 \cup U_0$ | |
|---|---|
| really possible individuals: $U_1$ | purely possible individuals: $U_0$ |

The objects of the domain are conceived as complete objects: that is, they are determined in their every attribute (property or relation). The elements of U are denoted in the semantic metalanguage by $\overline{x}$, $\overline{y}$, $\overline{z}$... . We assume, however, that for every individual of the domain there is at least one name in the PIE-language. And for the sake of convenience, we assume that it is the individual variables, when they are not quantified, that perform the function of individual names, i.e. constants. In this way, the individuals of the domain can also denoted by the signs x, y, x, …, the which, once an interpretation or evaluation function I of the language has been established, stand for specific objects in the individual domain. It should be noted that the interpretation function for the individual names is independent of the worlds because, as we shall shortly see, individual names are taken to be rigid designators. At this point, we can formally express the requirement of the completeness of possible objects as follows: for every individual denoted by x, and for every attribute denoted by $\alpha(x)$, either $\alpha(x)$ or $\neg\alpha(x)$ is the case.

***P*** is the set of the attributes (properties and relations) defined on U. Given the modal context, the attributes can be understood either intensionally or extensionally. The intension is given as a function which establishes the extension of the attribute for every world. In our semantic apparatus, however, it is important to establish that for all non-existential attributes, i.e. those other than the property of existence, the intension of the attribute fixes the same extension in each world. In other words, the essential attributes are conceived in rigid manner. The reason for this is that the individuals of U are possible objects, and possible objects do not vary with respect to non-existential attributes but do so solely because they exist or otherwise in a world – that is, because they are actualized or otherwise in that world. Attributes will be indicated in the semantic metalanguage by means of the signs $\overline{P}_i^n$ where *i* is the index of the order of the attribute and *n* is the number of its places. Of course, $\overline{P}_i^n \subseteq U^n$, i.e. $\overline{P}_i^n \in \wp(U^n)$. However, it is assumed that that attributes can also be nominated by means of the predicative constants of the PIE-language. In accordance with the rigid character of the non-existential attributes, the interpretation function for non-existential predicates is independent of the worlds, so that also the non-existential predicates are rigid designators. Of particular importance, in the set ***P***, are the attributes corresponding to the predicates of identity and p-identity: = and ≈. The attribute of existence will be treated separately, given that it is the only attribute understood in non-rigid manner.

*1.2.1. The identity predicate (=)*

The identity x = y states that the individual denoted by x and the individual denoted by y is the same individual in a particular world. The relation corresponding to the identity predicate = is $\overline{=}$. However, since $\overline{=}$ is the interpretation of =, it can also be indicated as the value of the interpretation function I(=). Thus defined, I(=) divides the domain of the language into equivalence classes which comprise a single element.

*1.2.2. The coincidence (p-identity) predicate (≈)*

Given the generality of the notion of possible object introduced in section 1.1., also possible individuals are the attributive variants of the same individuals, for example Socrates seated and Socrates standing. By hypothesis, also the names of these possibles exist. For example, let x be the name for Socrates seated and y the name for Socrates standing. Yet an identity relation cannot be predicated between x and y. It cannot be predicated because the notion obeys the law of substitution with identity (Leibniz's law of the indiscernibility of identicals), as a consequence of which the attributes of I(x) must also pertain to I(y); but this is impossible. It is true that Socrates standing is the same Socrates who, in another world, is seated. But precisely so: in another world. In the first world, Socrates standing is identical to the Socrates who will *become* seated in the second, but it is not identical to Socrates seated. However, I(x) is not extraneous to (completely independent of) I(y). Between I(x) and I(y) there is an identity relation *mediated* by the notion of possible world. Socrates seated is the same Socrates who, in a world alternative to the actual one, is standing. This mediated identity relation we shall call the *coincidence* relation (between possibles) or the *p-identity* relation, and we shall denote it with the sign $\overline{\approx}$. Of course, also with regard to the coincidence relation, this is the interpretation of coincidence predicate ≈, so that $\overline{\approx}$ can also be designated as the value of I(≈). Thus $x \approx y$ signifies that I(x) is a possible actualizable in some other world like I(y). Henceforth we shall also say that I(x) and I(y) represent two different individuations of the same individual. Unlike I(=), the coincidence relation I(≈) partitions the domain of the model into equivalence classes which comprise several elements. Each of these classes contains all and only the ontologically coincident possibles. The two types of relation are connected by the fact that an individual, in that it is always identical to itself, coincides with itself in every world, so that $x=y$ entails $x \approx y$. Finally, the distinction between identity and coincidence could be avoided if names were conceived as world-indexed descriptions. In this case, in fact, the difference between the two relations would disappear because the notion of possible world could no longer mediate between them. It would thus hold that the object Socrates that is seated-in-*u* is the same object Socrates that is standing-in-*v*, which is isomorphous to the morning star is the evening star. In what follows, however, we shall assume that individual names, like attributes in general, are not world-indexed.

### *1.3. Existence predicate* E

#### *1.3.1. Difference between essential predicates and the existence predicate*

The basic idea of the semantics presented here is that possible objects are the same in all worlds: what varies from world to world is only the extension of the existence predicate. This presupposes the distinction – of Kantian origin – between essential predicates, real in Kant's terminology, and the existence predicate, a non real predicate. A real predicate differentiates the individual which is predicated as to its concept: if we predicate being-a-lamp of an object, the concept of the object is therefore determined and differentiated from other possible concepts; likewise, if we predicate being-switched-on of a lamp, the concept of this object is correspondingly determined and differentiated. Conversely, the existence predicate does not differentiate the concept of the object being predicated: if we predicate being-existent of the lamp, we do not further determine the concept of the lamp. The existence predicate therefore says, not whether the object is determined in one way or another, but whether it is in one world or another. Consequently, an object may be existent in one world and non-existent in an alternative possible world, although identical with respect to the concept.

*1.3.2. The intension of the predicate* E

As said, it is a fundamental assumption of the PIE semantics that the objectual domain is constant for all possible worlds. However, *not the same possibles exist* in each of the possible worlds. Thus, the possible denoted by x exists in world u if and only if E(x) is true at u. This can also be expressed by saying that the extension of E varies from world to world, i.e. that the extension of E is the function $\boldsymbol{E} : W \mapsto \wp(U)$, and that this function is not constant. ***E*** must fulfil three requisites.

*1.3.3. Requisites of the **E** function*

It must first satisfy the following condition:

(1) Existence condition:

$$(\text{om } u)(\varnothing \neq \boldsymbol{E}(u) \subseteq U_1)$$

The existence condition ensures that the extension of the existence predicate is never (in any of the possible worlds) empty. This means that, for every world, at least one possible must be actualized. The rationale of this condition is obvious. A possible world is a really possible alternative to the actual world, which could not be if none of the possible objects of the world were existent in it. In other words, the real possibility of a world can only be founded on individuals that are really possible and, therefore, existent in it; conversely, a world founded on purely thinkable individuals is only an imaginary world.

(2) Coherence condition:

$$\boldsymbol{E}(u) \subseteq S[\approx U_1]$$

Where:

(i) $[\approx U_1]$ is the set of equivalence classes determined by the relation $\approx$.
(ii) $S[\approx U_1]$ is a selection set with respect to $[\approx U_1]$, a set that includes only one element for each of the equivalence classes determined by the coincidence relation.

The coherence condition establishes that every possible world only includes as existent one single individual of each equivalence class. This is entailed by the definition of the existence predicate and of the identity and coincidence relations. In fact, the property of existing in a world cannot be predicated of distinct possible coincident objects: if it could, a contradiction would immediately arise, because the same individual would exist in the same world in two different individuations, and would therefore be determined by two different sets of attributes; the same lamp would be both switched on and switched off in the same possible world. Consequently, an individual can possess different individuations, but only in different worlds, so that two different individuations of the same individual cannot be actual in the same world.

(3) Limited condition of exhaustiveness:

$$(\mathrm{om}\ \overline{x} \in U_1)(\mathrm{ex}\ u)(\overline{x} \in \boldsymbol{E}(u))$$

$\boldsymbol{E}(u)$ is such that, for every element $\overline{x}$ of $U_1$, and therefore for every really possible individual, there exists a possible world in which $\overline{x}$ exist. $\boldsymbol{E}(u)$ satisfies the limited – not general – condition of exhaustiveness because it is not possible to say that for every element $\overline{x}$ of U there exists a possible world in which $\overline{x}$ exist. Included in U, in fact, are purely thinkable individuals, which cannot be actual in any possible world. The exhaustiveness condition is not generally satisfied because possible existence coincides with real possibility, not with pure analytical possibility. If possible existence coincided with analytical possibility, one could correctly assume, for every pure possible (= non-contradictory) object, a pure possible (= non-contradictory) world in which the object is actualized. In fact, a non-contradictory world is a world that comprises non-contradictory objects, and if all non-contradictory worlds are possible worlds, all non-contradictory objects must be included in a world of this type; otherwise, not all non-contradictory worlds would be possible. However, once it has been posited that possible existence coincides with real possibility, not with purely analytical possibility, and that possible worlds are all really possible worlds, and not just analytically or logically possible ones, we may conclude that it may happen that a non-contradictory object is not actualized in any world.

Exemplification of the conditions introduced

$$U_1 = \{x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3\} \neq \varnothing$$

$$U = \{x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3\} \cup U_0$$

Assume that $x_1$, $x_2$, $x_3$ are three ultimate individuations of a possible object, so that the coincidence relation $x_1 \approx x_2 \approx x_3$ holds. Assume the same of the individuations $y_1$, $y_2$, $y_3$ and $z_1$, $z_2$, $z_3$. So three equivalence classes are determined in relation to three co-possible objects:

$[\approx x_1] = \{x_1, x_2, x_3\}$
$[\approx y_1] = \{y_1, y_2, y_3\}$
$[\approx z_1] = \{z_1, z_2, z_3\}$

$[\approx U_1] = \{[\approx x_1], [\approx y_1], [\approx z_1]\}$

The selection sets are in $[\approx x_1] \times [\approx y_1] \times [\approx z_1]$:

| | | |
|---|---|---|
| $\{x_1, y_1, z_1\}$ | $\{x_2, y_1, z_1\}$ | $\{x_3, y_1, z_1\}$ |
| $\{x_1, y_1, z_2\}$ | $\{x_2, y_1, z_2\}$ | $\{x_3, y_1, z_2\}$ |
| $\{x_1, y_1, z_3\}$ | $\{x_2, y_1, z_3\}$ | $\{x_3, y_1, z_3\}$ |
| $\{x_1, y_2, z_1\}$ | $\{x_2, y_2, z_1\}$ | $\{x_3, y_2, z_1\}$ |
| $\{x_1, y_2, z_2\}$ | $\{x_2, y_2, z_2\}$ | $\{x_3, y_2, z_2\}$ |
| $\{x_1, y_2, z_3\}$ | $\{x_2, y_2, z_3\}$ | $\{x_3, y_2, z_3\}$ |
| $\{x_1, y_3, z_1\}$ | $\{x_2, y_3, z_1\}$ | $\{x_3, y_3, z_1\}$ |
| $\{x_1, y_3, z_2\}$ | $\{x_2, y_3, z_2\}$ | $\{x_3, y_3, z_2\}$ |
| $\{x_1, y_3, z_3\}$ | $\{x_2, y_3, z_3\}$ | $\{x_3, y_3, z_3\}$ |

In each of the possible worlds of the structure constituted by domain U thus defined, the extension of the existence predicate is given by a sub-set of one of the selection sets relative to $U_1$. And because of the existence condition the sub-set in question cannot be empty.

*2.1 Universal model*

The universal model consists of the universal frame plus the interpretation of the PIE language on the structure:

$$M = \langle F,I \rangle$$

Interpretation I is a function which associates elements of the modal structure with the signs of the PIE language. This association depends on the worlds only for the existence predicate. In fact, in accordance with the nature of the elements of the modal structure introduced above, only the existence predicate is conceived as a non rigid designator. The other designators – i.e. the individual variables, the non-existential predicates, the identity and p-identity relations

– are all rigid. Specifically, I:

(a) associates possible objects with the individual variables:

$$I(x) = \overline{x} \text{ (with } \overline{x} \in U)$$

Of course, the function could be introduced as a two-argument function I(x,u), where the second argument consists of signs for worlds. In this case, the rigidity of individual names could be explicitly expressed by the formula:

$$(\text{om } x)(\text{om } u, v)(I(x,u) = I(x,v)) \qquad (\mathbf{Rig1})$$

Above, when presenting the rigid nature of the individual designators, we referred to the nature of the elements of the frame. This is characterized by the fact that the domain of the worlds is identical, and it is constituted by the totality of possible individuals.

(b) associates attributes with the non-existential predicative constants:

$$I(P^n_i) = \overline{P}^n_i \text{ (with } \overline{P}^n_i \subseteq U^n)$$

Here, too, the function could be introduced as a two-argument function $I(P^n_i,u)$, where the second argument consists of a sign for worlds. In this case, the rigidity of the non-existential predicates could be explicitly expressed by the formula:

$$(\text{om } P^n_i)(\text{om } u, v)(I(P^n_i,u) = I(P^n_i,v)) \qquad (\mathbf{Rig2})$$

That the non-existential predicates are rigid is due to the fact that possible objects are invariably characterized by the same attributes in every possible world.

(c) associates the identity and p-identity relations with the signs = and ≈:

$$I(=) = \{<\overline{x}, \overline{y}> \mid \overline{x}\in U \text{ and } \overline{y}\in U \text{ and } \overline{x} \overline{=} \overline{y}\}$$

$$I(\approx) = \{<\overline{x}, \overline{y}> \mid \overline{x}\in U \text{ and } \overline{y}\in U \text{ and } \overline{x} \overline{\approx} \overline{y}\}$$

(d) associates the existence property with the predicate E:

$$I(E,u) = \boldsymbol{E}(u)$$

The existence predicate is the only non-rigid designator. It is not possible to assume that (om u, v)(I(E,u) = I(E,v)) in relation to the existence predicate because different possible worlds are distinct precisely because different possible individuals are actual in them. It has even been said that one possible world is not determined by anything other than the set of possible individuals that are actual in it. If one modifies the set of actual individuals, one modifies the possible world in question. Note also that the extension of E is defined internally to $U_1$, and that the individuals included in $U_0$ are therefore individuals that do not exist in any possible world. The exhaustiveness condition is true with respect to $U_1$, but not with respect to U in general.

*2.2. Other semantic notions*

*Reinterpretation*

a) Reinterpretation of I with respect to x on $\overline{x}$

$I_x^{\overline{x}} =_{def}$ interpretation that differs from I at most for the value associated with x and such that, for all u, $I_x^{\overline{x}}(x,u) = \overline{x}$

b) Reinterpretation of M with respect to x on $\overline{x}$

Let M be <F,I>. Then, $M_x^{\overline{x}} =_{def} \langle F, I_x^{\overline{x}} \rangle$.

*Truth*

Base:

| | | |
|---|---|---|
| $\alpha \equiv P_i^n x_1 \ldots x_n$ | $M \models_u P_i^n x_1 \ldots x_n$ | $\Leftrightarrow I(P_i^n,u)$ holds of $I(x_1,u) \ldots I(x_n,u)$ |
| | | $\Leftrightarrow \langle \overline{x}_1 \ldots \overline{x}_n \rangle \in \overline{P}_i^n$ |
| $\alpha \equiv Ex$ | $M \models_u Ex$ | $\Leftrightarrow I(E,u)$ holds of $I(x,u)$ |
| | | $\Leftrightarrow \overline{x} \in \boldsymbol{E}(u)$ |

Step:

| | | |
|---|---|---|
| $\alpha \equiv \beta \wedge \gamma$ | $M \models_u \beta \wedge \gamma$ | $\Leftrightarrow M \models_u \beta$ and $M \models_u \gamma$ |
| $\alpha \equiv \neg\beta$ | $M \models_u \neg\beta$ | $\Leftrightarrow M \not\models_u \beta$ |
| $\alpha \equiv \forall x\beta$ | $M \models_u \forall x\beta$ | $\Leftrightarrow$ (om $\overline{x}$)($M_x^{\overline{x}} \models_u \beta$) |

$\alpha \equiv \Box\beta$ M $|=_u \Box\beta$ <=> (om v)(M $|=_v \beta$)

*Coincidence*

$M \overset{\alpha}{=} M' =_{def}$ (om x)(free x in $\alpha$ => M(x) = M'(x))
(M coincides with M' with respect to the free variables in $\alpha$)

*2.3 Some semantic theorems*

*Coincidence Theorem*: $M \overset{\alpha}{=} M'$ => (M $|=_u \alpha$ <=> M' $|=_u \alpha$)

As usual.

*Substitution Theorem*: Leg $\alpha^{y}_{x}$ => ($M^{M(y)}_{x} |=_u \alpha$ <=> M $|=_u \alpha^{y}_{x}$)

As usual.

*2.4. Soundness Theorem*

To be demonstrated is the soundness of:

1.Symmetry of ≈: x≈y |– y≈x
2.Transitivity of ≈: x≈y ∧ y≈z |– x≈z
3. Rule =≈: x=y |– x≈y
4. Rule ≈=: Ex ∧ Ey ∧ x≈y |– x=y
5. Rules NE e NI

Ad 1-4
The justification of the properties of simmetry and transitivity is obvious. The rule =≈ is justified by the fact that the identity relation is a stronger relation than that of coincidence. In the usual interpretation x=y signifies that x and y are two different names of the same individual. One may also say that the individual denoted by x is identical to the individual denoted by y. The difference with respect to x≈y resides in the fact that the individual denoted by x is identical to the individual denoted by y already in the world of x and not just in another world, as in the case of x≈y. Hence the truth of x=y entails the truth of x≈y. By contrast, the rule Tr≈= finds its justification in the fact that if x and y are coincident existents, then, because of the coherence condition of ***E***, they can only be identical. If they were not, x

and y would denote individuals different in all possible worlds, so that it could not hold that x denotes in the world of x the same individual which y denotes in another world: that is, $x \approx y$ cannot hold. On the other hand, nor could $x \approx y$ and simultaneously $x \neq y$ hold, given that in this case x and y could not be coincident possibles, but rather distinct ones because they are both existent.

Ad NE:

H: $M \models_u \alpha$, E do not occur in $\alpha$ Dem: $M \models_u \Box\alpha$

The proof is inductive.

Base: $\alpha \equiv Px$

| | |
|---|---|
| $M \models_u Px$ | H |
| $I(x,u) \in I(P,u)$ | def. $\models$ |
| (om x)(om u,v)(I(x,u) = I(x,v)) | (**Rig1**) |
| (om P)(om u,v)(I(P,u) = I(P,v)) | (**Rig2**) |
| $I(x,v) \in I(P,v)$ | by substitution |
| (om v)($I(x,v) \in I(P,v)$) | Iom |
| (om v)($M \models_v Px$) | def. $\models$ |
| $M \models_u \Box Px$ | def. $\models$ |

Step: $\alpha \equiv \beta \wedge \gamma$

| | |
|---|---|
| $M \models_u \beta \wedge \gamma$ | H |
| $M \models_u \beta$ and $M \models_u \gamma$ | def. $\models$ |
| $M \models_u \Box\beta$ and $M \models_u \Box\gamma$ | IH |
| $M \models_u \Box\beta \wedge \Box\gamma$ | def. $\models$ |
| $M \models_u \Box(\beta \wedge \gamma)$ | by T1 |

Step: $\alpha \equiv \neg\beta$

| | |
|---|---|
| $M \models_u \neg\beta$ | H |
| $M \not\models_u \beta$ | def. $\models$ |
| $M \models_v \beta \Rightarrow M \models_v \Box\beta$ | IH |
| $M \models_v \beta \Rightarrow$ (om u)($M \models_u \beta$) | def. $\models$ |

| | |
|---|---|
| $M \models_v \beta \Rightarrow M \models_u \beta$ | Eom |
| $M \not\models_u \beta \Rightarrow M \not\models_v \beta$ | contraposition |
| $M \not\models_u \beta \Rightarrow (\text{om } v)(M \not\models_v \beta)$ | Iom |
| $(\text{om } v)(M \not\models_v \beta)$ | mp |
| $(\text{om } v)(M \models_v \neg\beta)$ | def. $\models$ |
| $M \models_u \Box\neg\beta$ | def. $\models$ |

The steps involving the quantifiers do not raise difficulties in principle, and can therefore be proved in similar manner to the previous ones.

ad NI: Dem: $M \models_u \Box\exists xEx$

| | |
|---|---|
| $\varnothing \neq I(E,u) \subseteq U_1$ | Existence Condition |
| $I(E,u) \cap U_1 \neq \varnothing$ | by $\varnothing \neq I(E,u) \subseteq U_1$ |
| $(\text{ex } \overline{x})(\overline{x} \in U_1 \text{ and } \overline{x} \in I(E,u))$ | by def. $\cap$ |
| $(\text{ex } \overline{x})(\overline{x} \in I(E,u))$ | E and |
| $M \models_u \exists xEx$ | by def. $\models$ |
| $(\text{om } u)(M \models_u \exists xEx)$ | Iom |
| $M \models_u \Box\exists xEx$ | def. $\models$ |

*2.4. Completeness Theorem*

A first difficulty is due to the fact that the PIE model is a universal model, while the proof of completeness relative to S5 is based on models with R, which is an equivalence relation, i.e. S5-models. Consequently, one must show that working with universal models or with S5-models is equivalent. For this purpose a number of preliminaries are required. It is first necessary to formulate the distinction between our universal model and the S5-model for PIE:

Def. 1: universal model for PIE

$M = \langle F,I \rangle = \langle W, R, \boldsymbol{U}, \boldsymbol{E}, I \rangle$, where R is the universal relation defined on W: $R = W \times W$

Def. 2: S5-model for PIE

$M^E = \langle F^E,I \rangle = \langle W, R^E, \boldsymbol{U}, \boldsymbol{E}, I \rangle$, where $R^E$ is an equivalence relation defined on W

Def. 3: Logical consequence with respect to universal models

$X \Vdash_R \alpha \iff (\text{om} <W, R, \boldsymbol{U}, \boldsymbol{E}>)(\text{om I})(\text{om u})(M \models_u X \Rightarrow M \models_u \alpha)$

Def. 4: Validity with respect to universal models

$\Vdash_R \alpha \iff (\text{om} <W, R, \boldsymbol{U}, \boldsymbol{E}>)(\text{om I})(\text{om u})(M \models_u \alpha)$

Def. 5: Logical consequence with respect to S5-models

$X \Vdash_R^E \alpha \iff (\text{om} <W, R^E, \boldsymbol{U}, \boldsymbol{E}>)(\text{om I})(\text{om u})(M^E \models_u X \Rightarrow M^E \models_u \alpha)$

Def. 6: Validity with respect to S5-models

$\Vdash_R^E \alpha \iff (\text{om} <W, R^E, \boldsymbol{U}, \boldsymbol{E}>)(\text{om I})(\text{om u})(M^E \models_u \alpha)$

Theorem 1: $\Vdash_R \alpha \iff \Vdash_R^E \alpha$

Proof: (a) <=: immediate
(b): => H: $(\text{om M})(\text{om u})(M \models_u \alpha)$ Dem: $(\text{om } M^E)(\text{om u})(M^E \models_u \alpha)$
or H: $(\text{ex } M^E)(\text{ex u})((M^E \not\models_u \alpha)$ Dem: $(\text{ex M})(\text{ex u})((M \not\models_u \alpha)$

However, in order to obtain the result it is sufficient to reduce $M^E$ to M, keeping the equivalence class to which u belongs fixed. For the falseness of $\alpha$ depends solely on this class.

Corollary: $(X \Vdash_R^E \alpha \Rightarrow X \vdash_{PIE} \alpha) \Rightarrow (X \Vdash_R \alpha \Rightarrow X \vdash_{PIE} \alpha)$

In conclusion, the completeness of PIE with respect to the universal models ($X \Vdash_R \alpha \Rightarrow X \vdash_{PIE} \alpha$) is obtained from that with respect to the S5-models ($X \Vdash_R^E \alpha \Rightarrow X \vdash_{PIE} \alpha$). To this end it suffices to prove the following theorem:

Theorem 2: PIE-Cons X => $R^E$-Sod X, where :

(1) PIE- Cons X $=_{def.}$ $X \nvdash_{PIE} \perp$

(2) $R^E$-Sod X $=_{def.}$ $(\text{ex } M^E)(\text{ex u})(M^E \models_u X)$

The proof is based on construction of the canonical model for PIE, showing that a world that satisfies X exists in it, and that it is effectively a model for PIE. The original part of the proof is the third one, where it must be shown that the canonical model satisfies all the rules or axioms typical of PIE. The construction of the canonical model is the standard one. Let $L(PIE^{+})$ be a suitable linguistic extension of the language of PIE, so that it is possible to use Henkin's method. Let the class of all maximal and ω-complete sets of formulae of $L(PIE^{+})$ be constructed in the usual manner. Let the variables belonging to $L(PIE^{+})$ be divided into equivalence classes with respect to the identity relations which hold in the maximal and ω-complete sets of formulae of $L(PIE^{+})$. Let for example [=x] be one of these classes: it is constituted by all the variables y such that the relation x=y holds in one and thus in all the maximal and ω-complete sets of formulae belonging to $L(PIE^{+})$. Thus:

$W_C$ = class of the maximal and ω-complete PIE sets
$uR_Cv \Leftrightarrow (\text{om } \alpha)(\Box\alpha\in u \Rightarrow \alpha\in v)$
$U_c$ = set of equivalence classes $[=x_i]$ of variables belonging to $L(PIE^{+})$

The other essential elements of the canonical structure, namely $I_C(x_i,u)$, $I_C(P^n_i,u)$, $I_C(=,u)$, $I_C(\approx,u)$, $I_C(E,u)$ are defined so that they ensure the following clauses of the interpretation function:

(i) $I_C(x_i,u) = [=x_i]$, where $[=x_i]$ is the equivalence class of $x_i$[4]
(ii) $I_C(P^n_i,u)$ holds of $I_C(x_1,u)...I_C(x_n,u) \Leftrightarrow P^n_i\, x_1...x_n \in u$
(iii) $I_C(=,u)$ holds of $I_C(x_1,u)$ and $I_C(x_2,u) \Leftrightarrow x_2 = x_1 \in u$
(iv) $I_C(\approx,u)$ holds of $I_C(x_1,u)$ and $I_C(x_2,u) \Leftrightarrow x_2 \approx x_1 \in u$
(v)$I_C(E,u)$ holds of $I_c(x, u) \Leftrightarrow Ex \in u$

At this point, the proof of completeness divides into two parts.

(1) PIE-Cons X => (ex u)($M_C \models_u X$)

The proof is the classic one. It is based on the fact that, given the PIE-consistency of X, there exists an maximal and ω-complete extension of it, by a generalization of Lindenbaum's

[4] Note that it is also possible to avoid introducing the set of the equivalence classes of variables and to simply assert $I_C(x_i,u) = x_i$. However, in this case = is not interpreted in the standard manner. The identity-relation is to be treated in the same way as any relational predicate.

lemma. Therefore, given the definition of the canonical model, this extension is an element of $W_C$ and (ex u)($M_C$ $|=_u$ X).[5]

(2) $M_C$ is a model for PIE

To be proved is that $M_C$ possesses all the structural properties of any model for PIE. Now, a model for PIE is a model based on a structure for S5, i.e. a structure with an equivalence relation R. Moreover, the model is structurally rigid:

(i) $U_C$ is constant.
(ii) The individual names are rigid designators.
(iii) All the non-existential predicators are also rigid designators.
(iv) The two signs of identity are rigid designators.

Ad (i): $U_C$ is constant in that it coincides with the set of equivalence classes $[=x_i]$ of variables belonging to L(PIE+).
Ad (ii): The individual names are rigid designators because $I_C(x_i,u) = [=x_i]$.
Ad (iii): To show that the non-existential predicates are rigid designators, one must demonstrate that: (om u,v)($I_C(P^n_i,u)=I_C(P^n_i,v)$). To this end it is sufficient to show: (om u,v)($uR_Cv \Rightarrow (P^n_i x_1...x_n \in u \Leftrightarrow P^n_i x_1...x_n \in v)$).

Proof:

H: $uR_Cv$, $P^n_i x_1...x_n \in u$ Dem: $P^n_i x_1...x_n \in v$

| | |
|---|---|
| $P^n_i x_1...x_n \in u$ | H |
| $u \vdash P^n_i x_1...x_n$ | def. $\vdash$ |
| $u \vdash \Box P^n_i x_1...x_n$ | by NE |
| $\Box P^n_i x_1...x_n \in u$ | by closure of u |
| $uR_Cv$ | H |
| $P^n_i x_1...x_n \in v$ | by def. $R_C$ |

The other implication follows from the symmetry of $R_C$.

[5] For details see Hughes & Cresswell (1996), pp. 256-62.

Ad (iv): The rigidity of = and ≈ results from the fact that the identity and p-identity relations are necessary. This, for that matter, is already comprised in the fact that non-existential essential predicates are concerned.

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